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A new integer solution approach for fractional linear programming problem

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Abstract In mathematical programming different types of cuts have been developed in the past to get an integer value of the decision variables. In this paper, we have developed a new integer cut for getting an integer solution of the fractional linear programming problem (FLPP). This technique allows the decision-maker to solve the formulated FLPP to be conveniently using the Branch and Bound approach in order to achieve the optimum final solution. The process of development of the integer cut is shown with sufficient detail and a numerical illustration is used for clarification purpose.

Keywords Fractional linear programming problem; NAZ-CUT; Branch and bound

1. Introduction

Integer arises when a linear fractional function has to be maximized on a convex constraint polyhedron X. Consider a maximization linear FLPP may be stated as:

$$Max Q(x) = \frac{p(x)}{d(x)} = \frac{p^T x + \alpha}{d^T x + \beta}$$

s.t. $Ax \le \underline{b}$

$\underline{x} \ge 0$ and integer

Where x, p, d are $n \times 1$ vector, b is an $m \times 1$ vector and α, β are the scalar constants. It is assumed that the constraint $S = \{x: Ax \le b, x \ge 0\}$ is non-empty and bounded with $p(x) > 0$.

In the recent years, FLPP has received a lot of attention. Now a day many scientists and researchers uses numerous methodologies to solve the FLPP in different fields and applications. There are several realistic optimization problem causing this interest, where the objectives are quotient numbers of two functions, and used to obtain the maximum result-cost ratio, which is used for calculating the highest efficiency ratio. FLPP applications include the optimization of revenue/time, revenue/cost, efficiency/employee ratios, current assets/production, and cost-time ratios, etc. FLPPs can be used to address assignment problem, carriage problem, inventory issue, delivery problem, tree span minimum ratio problem, max route issue, and many more. The field of FLPP, largely developed by Hungarian Mathematician, Martors in 1960, is considered with the problem of optimization. Charnes and Cooper (1962) demonstrated that the initial FLPP can be easily transformed into LPP by basic conversion method, and this methodology is very helpful, as most conceptual results produced in LP could be generalised reasonably easily to solve the FLPP. Some recent work on the use of FLPP includes, Raina *et al*., (2018) used FLPP to formulate the transportation problem with discount cost function; similarly, Gupta *et al*., (2018) formulated capacitated transportation problem with multi-choices parameter as a FLPP; Ali *et al*., (2019) formulated the inventory model of ordering quantity as a FLPP and solved it using intuitionistic fuzzy goal programming; Coa (2020) formulated green supplier model as a FLPP and used fugitive measurement to determine the coefficient of relative proximity of green suppliers between intervals.

In most of the practical situations the values of the decision variables are required to be integer one. Integer programming in which all the variables are limited to a value of 0 or 1 is referred to as 0-1 integer or binary programming. Integer programming applications can be seen in a variety of fields such as fixed-capacitated problems, job or machine scheduling problems and many more, and along with a broad spectrum of real-life predicament problems can be presented as integer FLPP in operations, banking, social economics and philosophy. The general integer cutting plane algorithm initially uses the authenticity constraints on the variables and resolves the resultant LPP in a simpler way. Gomory (1958) developed the very first finite integer cut algorithm for getting the solution of integer LPPs. They demonstrated that, instead, a convex polyhedra can be specified as the conjunction of a limited number of partial spaces or convex framework plus the cylindrical hull of a certain number of dimensions or points. Based on this conjectural outcome, Gomory derived a “cutting plane” algorithm for integer LPPs. Dantzig (1963) also gave a new integer cutting plane technique for getting an integer value of the decision variable and, further this study was extended by others who worked on this topic are Glover(1965), Bowman(1970), Young(1971) and Balas (1973) etc. Combinatorial major problems, such as the problem of the budgeting of knapsack resources, the location of warehouses, traveling salesperson's problem, lower cost and network selection problems, channel and database problems such as , problems with flows, problems with matching solutions, weighted matching problems, problems with network coverage, and several scheduling issues can be addressed as integer FLPP. Integer FLPP have been solved by many authors in the past, Puri and Swarup (1965) suggested the extreme point mathematical programming technique for solving 0-1

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integer linear programming as well as integer FLPP, Grunspan and Thomas (1973) presented an algorithm for solving hyperbolic integer program which reduces to solving a sequence of linear integer problems, Agrawal (1976) provided an integer solution to the FLPP by using the branch and bound technique. Hiroaki (1976) gave a primal cutting algorithm for solving an integer FLPP. Chandra and Chandramohan (1979) gave an improved technique based on branch and bound method for solving a mixed integer FLPP, and later on Chandra and Chandramohan (1980) suggested an advanced form of their previous work by considering some more limitation in the branch and bound method. For 0-1 FLPP with single quotient term in the objective function, Anzai (1985) and Thirwani and Arora (1997) solved fractional objective functions and suggested its integral optimal solution by using the ranking function approach. Sharma and Bansal (2011) showed an integer solution of fractional programming by using a simplex method, Nachammai and Thangaraj (2011) presented the solution of a fuzzy FLPP using metric distance ranking, they also proposed the solution of a problem by considering fuzzy variables (Nachammai and Thangaraj, (2012)). Seerengasamy and Jeyaraman (2013) proposed a new approach to solve 0-1 Integer FLPP with the help of θ matrix function. Arefin *et al*., (2013) considered the additive algorithm in order to solve an integer FLPP class of 0-1, in which all the coefficients of an objective function are of similarly characteristic. Gupta *et al*., (2017) developed an iterative method for solving FLPP based on Beal's method and concluded that in the least number of iterations the proposed strategy offers an optimal solution. Youness *et al*., (2014) proposed an algorithm in integrating the Taylor series approach with the Kuhn Tucker assumptions to address the fuzzy two-level, multifunctional integer programming problem and then applying Gomory cuts to achieve the integer solution. These types of problems have attracted considerable research and interest because of less computational time. These types of problems have attracted considerable research and interest because of less computational time.

In this paper, we uses the NAZ-CUT developed by Bari and Ahmad (2003) for getting the optimum integer solution which is derived by finding the perpendicular distance from the integer points, which are inside the feasible region. We first find a continuous optimum solution by using simplex method or column simplex method then derive the NAZ-CUT by using the non-integer variables and add it to the constraints. After adding the NAZ-CUT the problem is then solved by branch and bound method. The additional secondary constraint (NAZ-CUT) with some key limits cuts off the infeasible space of the continuous solution. This cut off gives an ideal integer end point for the optimum existing convex solution. This means that after adding the cut, the current linear FLPP is now all integer FLPP. This also means that all NAZ-CUTs must proceed through at least one integer point which may or may be feasible. This cuts have been planned to dramatically reduce the total number of integer solutions in the corresponding feasible area. A numerical illustration is given to help explain the solution process of the method used.

The paper has been organized as follows: In section 2, the necessary NAZ cut for FLPP has been developed; section 3 contains algorithm for solving the problem. In section 4, the numerical illustration was demonstrated with a FLPP. Finally, in section 5, review of the solution and final conclusions are discussed.

2. Integer cut for solving FLPP

Given the objective function of FLPP (Bajalinov, 2003; Stancu-Minasian, 2012):

$$Q(x) = \frac{p(x)}{d(x)} = \frac{\sum_{j=1}^{n} p_j x_j + \alpha}{\sum_{j=1}^{n} d_j x_j + \beta}$$

which must be maximized (or minimized) subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_{i,} \qquad i = 1,2,\ldots m_1,$$

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_{i,} \qquad i = m_1 + 1, m_1 + 2, \ldots\ldots, m_{2,}$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_{i,} \qquad i = m_2 + 1, m_2 + 2, \ldots\ldots, m,$$

$$x_j \ge 0, \qquad j = 1,2,\ldots\ldots\ldots, n_{1,}$$

$$x_j \text{ - integer} \qquad j = 1,2,\ldots\ldots\ldots, n_{1,}$$

Where $m_1 \le m_2, n_1 \le n$. Here and in what follows we suppose that $d(x) \ne 0, \forall x = (x_1, x_2, \dots, x_n) \in S$,where S denotes a feasible set or set of feasible solution defined by constraints. Because denominator $d(x) \ne 0, \forall x \in S$ without loss of generality we can assume that $d(x) > 0, \forall x \in S$.In this case $d(x) < 0$ we can multiply numerator $p(x)$ and denominator $d(x)$ of objective function $Q(x)$ with (-1)We consider a pure integer FLPP in which all parameters are assumed to be integer. We start by solving the FLPP-relaxation to get a lower bound for the minimum objective value. First we find FLPP relaxation and let it be x^*. If x^* is all integer, then the problem is solved. Let the k^{th}component of x^*be non-integer with value $x^k = a_k^*$. The nearest integer value to x^k are $x_1^k = [a_k^*]$ and $a_2^k = [a_k^*] + 1 = \langle a_k^* \rangle, k = 1,2, \dots, n$ .Where $[t]$ is the largest integer less than or equal to t and $\langle t \rangle$ is the smallest integer greater than or equal to t. With such bifercations we can find all the 2^n integer points in the surrounding of the non-integer solution x^*. (e.g. in case of two variable problem if x^*. =(1,5,2,2). then there will be $2^2 = 4$ integer points $(1,2), (1,3), (2,2), (2,3)$ around this x^*. Denote the set of indices of these 2^n points by T. Let the objective value at x^*be Q^*. Thus the objective function level plane at x^*will be $\frac{px^*}{dx^*} = Q^*$

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Now we find the perpendicular distance d_i from the surrounding points, to the objective plane by using the formula

$$d_i = \frac{\dfrac{px^* + \alpha}{dx^* + \beta} - \dfrac{px_i^o + \alpha}{dx_i^o + \beta}}{\sqrt{\sum_{j=1}^{n} p_j^2 \Big/ \sum_{j=1}^{n} d_j^2}}, i \in T$$

Where x_i^0 is an integer point around x^*. Now we search for the point x_i^0 , which has a minimum distance from the objective function hyper plane. Obviously the negative distances and the distances from the infeasible points should be omitted. We choose the minimum positive distance only from the points, which are feasible. Let S be the set of indices $i \in T$ for which $x_i^0$are feasible. Let, $x_i^0 = \{x_k^0 / d_k = min_{i \in S} d_i\}$, we will introduce the NAZ CUT at x^0 by using the following relationship

$$(p\underline{x} + \alpha) + (d\,\underline{x} + \beta) \geq (px^0 + \alpha) + (dx^o + \beta)$$

3. Algorithm for solving the problem

The step wise step algorithm for solving the problem is as follows:

STEP 1: Using either simplex method or column simplex method to solve FLPP.

STEP 2: If the obtained solution is integer, then STOP. If not, round off the non-integer solution to the closest integer.

STEP 3: Find the minimum perpendicular distance from the point of integer inside the feasible restrictions area. Derive NAZ- CUT passing through this point and parallel to any of the constraints.

STEP 4: Use branch and bound or cutting plane method for getting the optimum integer solution.

4. Numerical

This section presents an example to demonstrate the used method Subject to constraints

$$5x_1 + 3x_2 \leq 6$$
$$7x_1 + x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

Where

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$\underline{c}_p = (2,1)$, $\underline{x} = (x_1, x_2)$

$\underline{c}_d = (7,1)$, $\underline{x}_s = (x_3, x_4)$

$A = \begin{pmatrix} 5 & 3 \\ 7 & 1 \end{pmatrix}$

The starting column simplex tableau is given as

Table 1

		x_1	x_2	
Q	0	-2	-1	
p	0	-2	-1	
d	6	-3	-1	
x_1	0	-1	0	-
x_2	0	0	-1	-
x_3	6	5	3	6/5
x_4	6	7	1	6/7 ←
		↑		

$\Delta_1 = -2 + 0\times 3 = -2$ $\Delta_2 = -1 + 0\times 1 = -1$

The most negative $\Delta_i = \Delta_1 = -2$ which correspond to $x_1 \Rightarrow x_1$.Will become basic and the column corresponding to x_1 will be the pivotal column. The minimum positive ratio of the present solution column to the pivotal column is attained corresponding to $x_4 \Rightarrow x_4$ row will be the pivotal row. $x_4$will become non basic and the element 7 at the intersection of the pivotal row and the pivotal column will be the pivotal element. The next iteration is obtained by

(1) Dividing the pivotal column by -7
(2) Applying the elimination procedure to transform the pivotal row (6, 7, 1) into (0,-1, 0).

The subsequent tableau of the column simplex method is:

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Table 2

		x_4	x_2	
Q	1/5	1/5	-3/5	
p	12/7	2/7	-5/7	
d	60/7	3/7	-4/7	
x_1	6/7	1/7	1/7	6/7
x_2	0	0	-1	-
x_3	12/7	-5/7	16/7	12/16 ←
x_4	0	-1	0	-

↑

$$\Delta_4 = \frac{2}{7} - \frac{1}{5} \times \frac{3}{7} = \frac{1}{5} \quad \Delta_2 = \frac{-5}{7} + \frac{1}{5} \times \frac{4}{7} = \frac{-3}{5}$$

The most negative $\Delta_i = \Delta_2 = \frac{-3}{5}$ which correspond to x_2.⇒ x_2Will become basic and the column corresponding to x_2 will be the pivotal column.The minimum positive ratio of the present solution column to the pivotal column is attained corresponding tox_3.⇒ the x_3 row will be the pivotal row. x_3 will become non basic and the element 16/7 at the intersection of the pivotal row and the pivotal column will be the pivotal element. The next iteration is obtained by

(1) Dividing the pivotal column by -16/7
(2) Applying the elimination procedure to transform the pivotal row $\left(\frac{12}{7}, \frac{-5}{7}, \frac{16}{7}\right)$ into $(0,0,-1)$

The subsequent tableau of the column simplex method is:

Table 3

		x_4	x_3
Q	1/4	0	1/4
p	9/4	1/16	5/16
d	9	1/4	1/4
x_1	3/4	3/16	-1/16
x_2	3/4	-5/16	7/16
x_3	0	0	-1
x_4	0	-1	0

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$$\Delta_4 = \frac{1}{16} - \frac{1}{4} \times \frac{1}{4} = 0 \ , \quad \Delta_3 = \frac{5}{16} - \frac{1}{4} \times \frac{1}{4} = \frac{1}{4}$$

Since all $\Delta_i \geq 0$, the optimal continuous solution is given by

$$x_1 = \frac{3}{4} \ x_2 = \frac{3}{4} \ Q = \frac{1}{4}$$

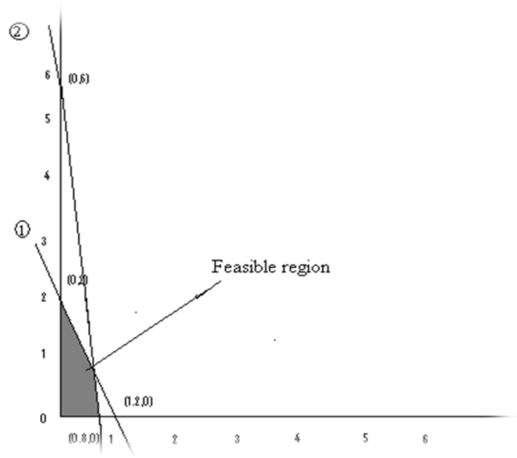

Figure 1. Graphical presentation of the FLPP without cut

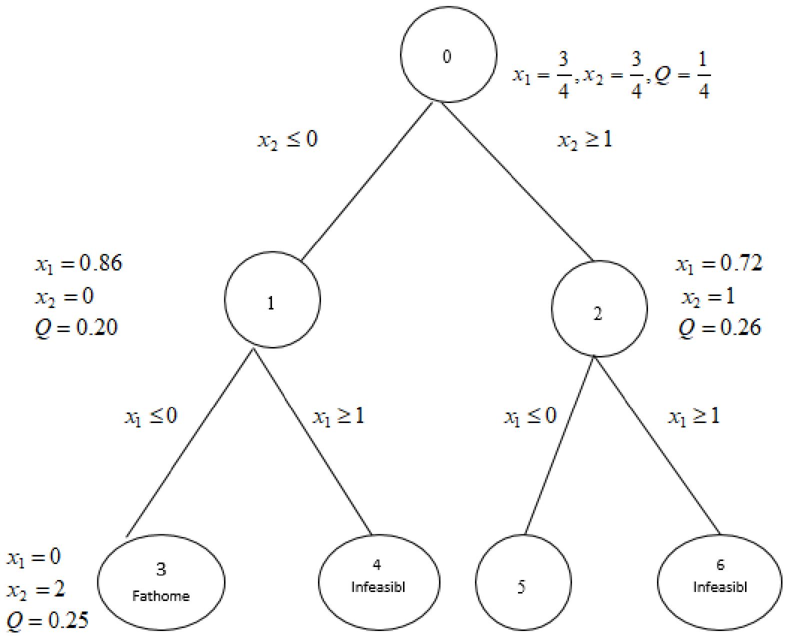

Figure 2. Branch and bound of the FLPP when NAZ CUT is not added

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But the value of x_1 and x_2 are not integer. So, we make them integer by using NAZ CUT. So we round off the integer solution to the nearest integer points (0,0), (0,1), (1,0),(1,1). Now use the distance formulation to measure the distance perpendicular for the points.

The distance from the point (0, 0) is $\frac{0.25-0}{\sqrt{5/10}}=0.35$

The distance from the point (0, 1) is $\frac{0.25-0.143}{\sqrt{5/10}}=2.463$

The distance from the point (1, 0) is $\frac{0.25-0.22}{\sqrt{5/10}}=0.04$

The distance from the point (1, 1) is $\frac{0.25-0.3}{\sqrt{5/10}}=-0.71$

We reject those points where the distance is negative and verify that the assumptions upon which distance is positive are fulfilled. If the assumptions are not fulfil then discard that point. Now we are left with only two points which is in the feasible region of which we select (0, 1) as (0, 0) does not effect the feasibility of the solution.

Now we derive NAZ CUT along with constraints passing through the integer point (0,1), we have

$$\left.\begin{array}{l} 5x_1 + 2x_2 \geq 25 \\ x_1 \geq 0 \\ x_1 \geq 1 \end{array}\right\} \quad (3)$$

Graphical presentation of FLPP when NAZ-CUT is added to the main constraints

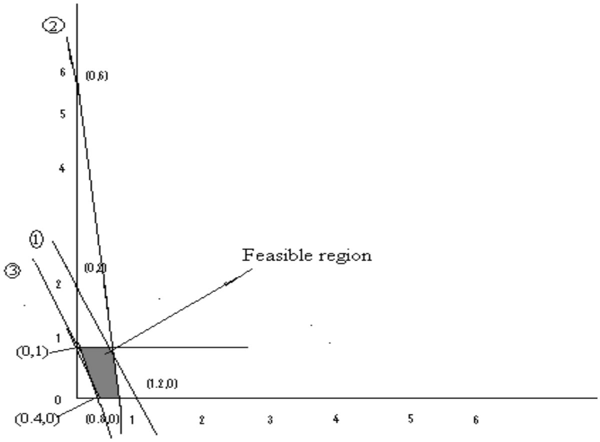

Figure 3. Graphical presentation of FLPP when NAZ cut is added

After adding the NAZ-CUT, the number of iteration is difficult to estimate by using the simplex method but being an exact method like branch and bound, the computational time is not polynomial bounded so the new problem after adding the cut is solved by using a branch and bound method which gives integer solution to the problem in a quick time (See Figure 3).

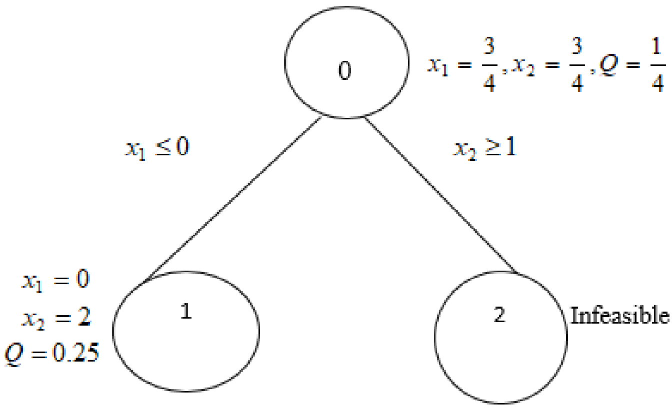

Figure 4. Branch and bound of the FLPP when NAZ CUT is added

By adding the NAZ-CUT to FLPP, we get the following integer solution:

$$x_1 = 0,\ x_2 = 2 \text{ and } Q = 1/4$$

5. Conclusion

In this paper, our main aim to develop the direct technique for solving the FLPP in fewer number of iterations. Here, we considered the NAZ-CUT proposed by Bari and Ahmad (2003) for getting an integer solution of a LPP. Most of the algorithms used to solve the

FLPP rely on the conventional basic technique. The algorithm that we suggested in this paper is based on the extension of NAZ-CUT which is used for solving the LPP. The key goal of creating this approach was that we didn't have to turn the FLPP into an LPP and it also helps to locate the feasible area using the NAZ-CUT into a series of feasible directions. Due to the simplicity of this method, anyone can conveniently use this approach to solve the FLPP to find the optimum solution. As this method consumes less time and it is very easy to understand and apply, it also reduces the complexity faces at the time of solving the problem by other integer cuts. Numerical results are often provided to show, that in minimum number of iterations with limited computing time algorithms can solve relatively large problems. In future, we will try to explore the new methodology for multi-objective FLPP quadratic FLPP, multi-objective quadratic FLPP.

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