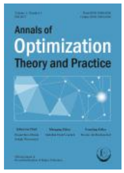
Annals of Optimization Theory and Practice

Volume 1, Number 1, 35-42
February 2018

DOI: 10.22121/aotp.2018.115787.1009



Optimal system and approximate solutions of the nonlinear filtration equation

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(Received: December 15, 2017 / Accepted: January 10, 2018)

Abstract In this paper, the problem of determining the most general Lie point approximate symmetries group for the nonlinear filtration equation with a small parameter is analyzed. By applying the basic Lie approximate symmetry method for the nonlinear filtration equation with a small parameter, the classical Lie point approximate symmetry operators are obtained. Also, the algebraic structure of the Lie algebra of approximate symmetries is discussed and an optimal system of one-dimensional subalgebras of the nonlinear filtration equation with a small parameter, symmetry algebra which creates the preliminary classification of group invariant solutions is constructed. Particularly, the Lie invariants as well as similarity reduced equations corresponding to infinitesimal symmetries and group invariant solutions associated to the symmetries are obtained.

Keywords Lie group analysis; Approximate symmetry; Optimal system; Invariant solution; Filtration equation.

1. Introduction

The classical Lie Symmetry method, originally introduced by Sophus Lie (1895). It was then popularized by Ovsiannikov (1982) and presented in a modern form using the jet space theory by Olver (1993). This method leads us to one-parameter group of transformations called classical symmetries that leaves the equation unchanged, and hence, they map the set of all solutions to itself. These symmetries are used to reduce the order of ordinary differential equations, or to reduce the number of independent variables of PDEs Camacho *et al.* (2011). The fact that symmetry reductions for many PDEs cannot be determined via the classical symmetry method, is the source of motivated to create several generalizations of the classical Lie group approach for symmetry reductions. Consequently, several alternative reduction methods have been proposed, such as LieBảcklund symmetry, nonclassical symmetry, potential symmetry, etc. Bluman and Kumei (1989) and Bluman *et al.* (2010). One of these techniques which is extremely applied particularly for nonlinear problems is perturbation analysis. It is worth mentioning that sometimes differential equations which appear in mathematical models

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are presented with terms involving a parameter called the perturbed term. Because of the instability of the Lie point symmetries with respect to perturbation of coefficients of differential equations, a new class of symmetries has been created for such equations, which are known as approximate symmetries. In the last century, in order to have the utmost result from the methods, combination of Lie symmetry method and perturbations are investigated and two different so-called approximate symmetry methods have been developed. The first method was presented by Baikov *et al.* (1989). The second approach was proposed by Fushchich and Shtelen (1989) and later was followed by Euler *et al.* (1992). This method is generally based on the perturbation of dependent variables. In Pakdemirli *et al.* (2004), a comprehensive comparison of these two methods is presented.

We will investigate the vector fields, approximate symmetry, symmetry reductions, optimal system and new exact solutions to the perturbed nonlinear filtration equation with a small parameter equation which describes a pressure distribution in a porous medium Alexandrova *et al.* (2014):

$$u_t = u_x^2 u_{xx} + \varepsilon u, \tag{1}$$

where $\varepsilon \ll 1$ is a small parameter, with the method of Baikov *et al.* (1989) and Ibragimov and Kovalev (2009).

This paper is organized as follows: In Section 2, we present approximate symmetry of the perturbed nonlinear filtration equation with a small parameter. Section 3, we present optimal system of the perturbed nonlinear filtration equation with a small parameter. In Section 4, is devoted to symmetry reductions of ordinary differential equations and the exact analytic solutions to the equation. Finally, the conclusions are presented by Section 5.

2. Approximate symmetry

In this paper the approximate equation $f \approx g$ means that $f(x.\varepsilon) = g(x.\varepsilon) + o(\varepsilon)$ and

$$F(z,\varepsilon) \approx F_0(z) + \varepsilon F_1(z) = u_t - u_x^2 u_{xx} - \varepsilon u.$$

Recall that the generator of an approximate transformation group admitted by Eq.(1) is written in the form of Bluman *et al.* (2010) and Gazizov (1996):

$$X = \tau(t,x,u,\varepsilon)\partial_t + \xi(t,x,u,\varepsilon)\partial_x + \eta(t,x,u,\varepsilon)\partial_u, \tag{2}$$

Where

$$\tau(t,x,u,\varepsilon) \approx \tau_0(t,x,u) + \varepsilon\tau_1(t,x,u)$$
$$\xi(t,x,u,\varepsilon) \approx \xi_0(t,x,u) + \varepsilon\xi_1(t,x,u)$$
$$\eta(t,x,u,\varepsilon) \approx \eta_0(t,x,u) + \varepsilon\eta_1(t,x,u)$$

It is convenient to identify X with its canonical representative: $X = X_0 + \varepsilon X_1$. If the vector field (2) generates an approximate symmetry of the equation (1), then X must satisfy the Lie approximate symmetry condition:

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$$\left[X^{(2)}F(z,\varepsilon)\right]_{F(z,\varepsilon)\approx 0}=o(\varepsilon), \tag{3}$$

Or

$$\left[X_0^{(2)}F_0(z)+\varepsilon(X_1^{(2)}F_0(z)+X_0^{(2)}F_1(z))\right]_{F(z,\varepsilon)\approx 0}=o(\varepsilon), \tag{4}$$

where $X^{(2)}$ denotes the 2-th order prolongation of X. Equation (3) is the determining equation for infinitesimal approximate symmetries. Solving system (4), we obtain

$$\begin{aligned} X_0 &= (c_1 t + c_2)\partial_t + (\tfrac{1}{4}(c_1+2c_3)x + c_5)\partial_x + (c_3 u + c_4)\partial_u, \\ X_1 &= (-c_1 t^2 + c_6 t + c_7)\partial_t + (\tfrac{1}{4}(c_6+2c_8)x + c_{10})\partial_x + ((c_1 t + c_8)u + c_4 t + c_9)\partial_u. \end{aligned} \tag{5}$$

Therefore

$$X=(c_1t+c_2+\varepsilon(-c_1t^2+c_6t+c_7))\partial_t+(\frac{1}{4}(c_1+2c_3)x+c_5+\varepsilon(\frac{1}{4}(c_6+2c_8)x+c_{10}))\partial_x+(c_3u+c_4+\varepsilon((c_1t+c_8)u+c_4t+c_9))\partial_u \tag{6}$$

Where $c_i. i = 1.2 \ldots .10$ are arbitrary constants. The infinitesimal approximate symmetries of Eq.(1) form the eight-dimensional approximate Lie algebra Fushchich and Shtelen (1989) spanned by the following independent operators:

$$\begin{aligned} &v_1=\partial_t,\ v_2=\partial_x,\ v_3=(\varepsilon t+1)\partial_u,\ v_4=(4t-4\varepsilon t^2)\partial_t+x\,\partial_x+4\varepsilon tu\,\partial_u \\ &v_5=x\,\partial_x+2u\,\partial_u,\ v_6=\varepsilon\partial_t,\ v_7=\varepsilon\partial_x,\ v_8=\varepsilon\partial_u, \end{aligned} \tag{7}$$

where their approximate commutator, evaluated in the first order of precision, is given in Table 1.

Table 1. Approximate commutators of approximate symmetry of Eq.(1).

$[v_i, v_j]$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	0	0	0	0	v_2	v_5	$2v_4$	v_7
v_2	0	0	0	0	0	0	v_5	v_6
v_3	0	0	0	$-v_5$	0	0	$-v_6$	0
v_4	0	0	v_5	0	0	0	0	0
v_5	$-v_2$	0	0	0	0	0	0	0
v_6	$-v_5$	0	0	0	0	0	0	0
v_7	$-2v_4$	$-v_5$	v_6	0	0	0	0	0
v_8	$-v_7$	$-v_6$	0	0	0	0	0	0

The approximate operator $X = X_0 + \varepsilon X_1$ generates the one-parameter approximate transformation group given by the following approximate exponential map Gazizov (1996):

$$\bar{x}^i=(1+\varepsilon \ll aX_0, aX_1 \gg)\exp(aX_0)(x^i), \qquad i=1,2,3.$$

Where $x^1 = t$, $x^2 = x$, $x^3 = u$ and

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$$\exp(aX_0) = 1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \dots,$$

and

$$<< aX_0, aX_1 >>= aX_1 + \frac{a^2}{2!}[X_0, X_1] + \frac{a^3}{3!}[X_0,[X_0, X_1]] + \dots$$

Furthermore, for Eq.(1), the one-parameter approximate transformation group G_i generated by the v_i for $i = 1.2. \dots .8$ are given in the followings:

$$\begin{aligned}
G_1: &\quad (t,x,u) \mapsto (t+a,x,u),\\
G_2: &\quad (t,x,u) \mapsto (t,x+a,u),\\
G_3: &\quad (t,x,u) \mapsto (t,x,u+a(1+\varepsilon t)),\\
G_4: &\quad (t,x,u) \mapsto (\tfrac{t}{\varepsilon t + e^{-4a}(1-\varepsilon t)}, xe^a, u(1-\varepsilon t+\varepsilon te^{4a})),\\
G_5: &\quad (t,x,u) \mapsto (t, xe^a, ue^{2a}),\\
G_6: &\quad (t,x,u) \mapsto (t+a\varepsilon,x,u),\\
G_7: &\quad (t,x,u) \mapsto (t,x+a\varepsilon,u),\\
G_8: &\quad (t,x,u) \mapsto (t,x,u+a\varepsilon),
\end{aligned} \tag{8}$$

Consequently, if $u = f(t.x)$ is a solution of the Eq.(1), so are the functions:

$$\begin{aligned}
&G_1(a).f(t,x) = f(t-a,x),\\
&G_2(a).f(t,x) = f(t,x-a),\\
&G_3(a).f(t,x) = a(1+\varepsilon t) + f(t,x),\\
&G_4(a).f(t,x) = \frac{f(\frac{t}{\varepsilon t+(1-\varepsilon t)e^{4a}}, \frac{x}{e^a})}{1-\varepsilon t+\varepsilon te^{-4a}},\\
&G_5(a).f(t,x) = e^{2a} f(t, xe^{-a}),\\
&G_6(a).f(t,x) = f(t-a\varepsilon,x),\\
&G_7(a).f(t,x) = f(t,x-a\varepsilon),\\
&G_8(a).f(t,x) = a\varepsilon + f(t,x).
\end{aligned} \tag{9}$$

3. Optimal system

As is well known, the theoretical Lie group method plays an important role in finding exact solutions and performing symmetry reductions of differential equations. Since any linear combination of infinitesimal generators is also an infinitesimal generator, there are always infinitely many different symmetry subgroups for the differential equation. So, a mean of determining which subgroups would give essentially different types of solutions is necessary and significant for a complete understanding of the invariant solutions. As any transformation in the full symmetry group maps a solution to another solution, it is sufficient to find invariant solutions which are not related by transformations in the full symmetry group, this has led to the concept of an optimal system. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially

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the same as the problem of classifying the orbits of the adjoint representation. This problem is attacked by the naive approach of taking a general element in the Lie algebra and subjecting it to various adjoint transformations so as to simplify it as much as possible. One of the applications of the adjoint representation is classifying group-invariant solutions Olver (1993).

The adjoint action is given by the Lie series:

$$Ad(\exp(\mu v_i)v_j) = v_j - \mu[v_i, v_j] + \frac{\mu^2}{2!}[v_i,[v_i,v_j]] - \ldots \tag{10}$$

Where $[v_i.v_j]$ is a commutator for the Lie algebra, μ is a parameter, and $i.j = 1.2.\ldots.8$

In Table 2 we give all the adjoint representations of the Lie group, with the $(i.j)$ the entry indicating $Ad(\exp(\mu v_i))v_j$.

Table 2. Adjoint representation generated by the basis approximate symmetries of the Lie algebra (7).

Ad	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	v_1	v_2	v_3	v_4	$v_5 - \mu v_2$	$v_6 - \mu v_5 + \frac{\mu^2}{2} v_2$	$v_7 - 2\mu v_4$	$v_8 - \mu v_7 + \mu^2 v_4$
v_2	v_1	v_2	v_3	v_4	v_5	v_6	$v_7 - \mu v_5$	$v_8 - \mu v_6$
v_3	v_1	v_2	v_3	$v_4 + \mu v_5$	v_5	v_6	$v_7 + \mu v_6$	v_8
v_4	v_1	v_2	$v_3 - \mu v_5$	v_4	v_5	v_6	v_7	v_8
v_5	$v_1 + \mu v_2$	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_6	$v_1 + \mu v_5$	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_7	$v_1 + 2\mu v_4$	$v_2 + \mu v_5$	$v_3 - \mu v_6$	v_4	v_5	v_6	v_7	v_8
v_8	$v_1 + \mu v_7$	$v_2 + \mu v_6$	v_3	v_4	v_5	v_6	v_7	v_8

An optimal system of one-dimensional approximate Lie algebras of the perturbed nonlinear filtration equation with a small parameter equation is provided by:

$$\begin{gathered} v_1, \quad av_1 + v_2, \quad av_1 + bv_2 + v_3, \quad \alpha v_1 + av_3 + bv_4 + v_5, \\ \alpha v_1 + av_3 + bv_4 + v_6, \quad \alpha v_1 + \beta v_3 + av_4 + v_7, \quad \alpha v_1 + av_3 + bv_4 + cv_6 + v_8, \end{gathered} \tag{11}$$

where $a.b.c$ and $\alpha.\beta \neq 0$ are arbitrary constants.

4. Symmetry Reductions

Reduction 1. Similarity variables related to the generator $v_1 = \partial_t$, are $u(t.x) = \emptyset(x)$, Substituting into Eq.(1), we reduce it to the following ODE:

$$\phi'^2\phi'' + \varepsilon\phi = 0, \tag{12}$$

where $\phi' = \frac{d\phi}{dx}$.

Reduction 2. In general, the traveling wave solutions to a PDE arise as special group-invariant solutions in which the group under consideration is a translational group on the

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space of independent variables. In the present case, we consider the generator $v_1 + \alpha v_2 (\alpha \neq 0)$, in which α is a fixed constant which determines the speed of the waves. Global invariants of this group are as follow,

$$u(t,x) = \phi(x - \alpha t), \tag{13}$$

Substituting it into Eq.(1), we find the reduced ordinary differential equation for the traveling wave solutions to be

$$\phi'^2 \phi'' + \alpha \phi' + \varepsilon \phi = 0, \tag{14}$$

where $\phi' = \frac{d\phi}{d\eta}$ $(\eta = x - \alpha t)$.

Reduction 3. Similarity variables related to the generator $v_5 = x\partial_x + 2u\partial_u$, are $u(t.x) = x^{-2}\phi(t)$, Substituting into Eq.(1), we reduce it to the following ODE:

$$\phi' - \varepsilon \phi + \frac{24}{x^8}\phi^3 = 0, \tag{15}$$

where $\phi' = \frac{d\phi}{dt}$, the solution of this equation is

$$\phi(t) = \pm \frac{x^4 \sqrt{\varepsilon(c\varepsilon x^8 e^{-2\varepsilon t} - 24)}}{c\varepsilon x^8 e^{-2\varepsilon t} - 24}, \tag{16}$$

where c is arbitrary constant, and consequently solutions for the Eq.(1) as follows:

$$u(t,x) = \pm \frac{x^2 \sqrt{-24\varepsilon}}{c\varepsilon x^8 e^{-2\varepsilon t} - 24}, \tag{17}$$

In Figure1 we plot solution (17) with $\varepsilon = -0.01$ and $c = 2$. In Figure 2 we plot solution (17) with $\varepsilon = -0.1$ and $c = 2$.

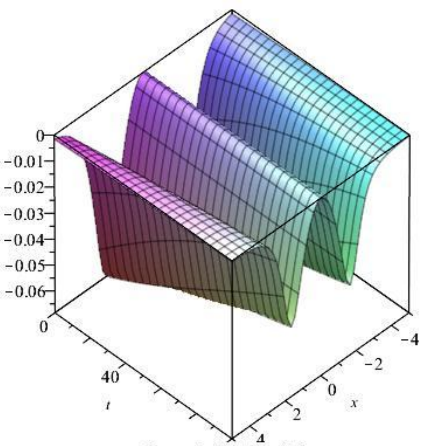

Figure 1. Solution (4.6)

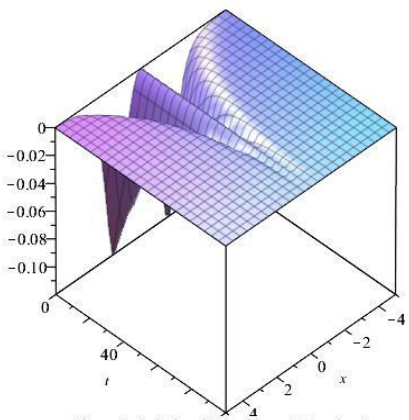

Figure 2. Solution (4.6) with ε=-0.1 and c=2

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Reduction 4. For linear combination $v_2 + v_3 = \partial_t + \partial_u + \varepsilon t \partial_u$ the invariants are $J_1 = (1+\varepsilon)t, J_2 = u - x(1+\varepsilon t)$ and $u = x(1+\varepsilon t) + \phi(t + \varepsilon t)$. Substituting into Eq.(1), we reduce it to the following ODE:

$$\phi' - \frac{\varepsilon}{1+\varepsilon}\phi = 0, \tag{18}$$

where $\phi' = \frac{d\phi}{dJ_1}$, the solution of this equation is

$$\phi(J_1) = ce^{\varepsilon t}, \tag{19}$$

where c is arbitrary constant, and consequently solutions for the Eq.(1) as follows:

$$u(t,x) = x(1+\varepsilon t) + ce^{\varepsilon t}. \tag{20}$$

5. Conclusions

In this paper, Lie approximate symmetry analysis was applied to study the nonlinear filtration equation with a small parameter. We obtained Lie approximate algebra, similarity reductions of this equation. Some of the group-invariant solutions to the Eq.(1) are considered based on the optimal system method. Then, we construct new analytical solutions with a small parameter to the Eq.(1).

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