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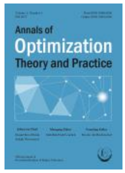

Evaluating efficiency and ranking of technical efficient and inefficient units

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Abstract Maintaining the efficiency score of efficient units and distinguishing between technical efficiency and efficiency are very essential topics in operations research. The Arash method is a new technique in Data Envelopment Analysis (DEA) for estimating the performance of units and ranking. The Arash method is based on additive DEA model (ADD). In this study we expand the Arash method by using facet analysis to modify the production possibility set (PPS). This modification avoids the effect of the weak part of PPS frontier which can propose a bias efficiency evaluation to DMUs placed on or compared with the weak part of the frontier. We integrated the attributes of the Arash method and modified variable return to scale model, consequently showing the true efficiency score and ranks of units associated with the weak part of the frontier, also identifying "technical efficient" and "inefficient" DMUs. The efficiency score of DMUs located at the strong part of the frontier remains the same, only those associated with the weak part are modified. A numerical example is used to show the effectiveness of the proposed model in comparison with the Arash method.

Keywords Efficiency; Ranking; Arash method; Facet analysis; Data Envelopment Analysis

1. Introduction

Performance evaluation and assessment of organizations efficiency and productivity is one of the fundamental aspects of economics and management. Also, identifying the true efficiency, productivity, quality of output, and true value of a business, cannot be overemphasized in the business world. As the industrial world continues to be competitive, organizations and businesses try to grow and achieve global dominance. For an organization to have a competitive edge, their subsidiaries or units need to perform

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efficiently. Therefore, performance evaluation is a necessary tool used to identify the strength and weaknesses of an organization or business.

Data Envelopment Analysis (DEA) is an important technique in evaluating performance of an organization or business. It is a performance measurement technique that has been successfully implemented in a wide range of areas to evaluate entities, comparable units or systems known as decision making units (DMUs) such as, healthcare Ibrahim and Daneshvar (2018), policy analysis Ibrahim *et al.* (2018), banking industry Jamshidi *et al.* (2019), Natural resource management Ibrahim *et al.* (2019) and Sanei and Hassasi (2018) etc. DEA was developed by Charnes *et al.* (1978) and was then extended Banker *et al.* (1984) to include variable return to scale. It is regarded as data-oriented because it affects performance evaluation and other interferences directly and with minimal assumptions. DEA has been widely accepted as an effective performance evaluation technique for measuring relative efficiency. This led to improved theoretical development and practical application in many fields Sanei and Hassasi (2018).

Evaluating DMUs in DEA has its limitations, one of which is ranking of DMUs, and ranking is an important issue in DEA studies. The efficiency score of the evaluated DMUs is from zero to one, with the efficient DMUs taking a score of one. A unique objective of DEA is to find the most efficient DMU among the homogenous evaluated DMUs. This proved difficult because multiple DMUs among the evaluated DMUs take a score of one, which leads researchers to develop methods of distinguishing or ranking the DMUs that are efficient after evaluation. Numerous papers have been published on how to rank both efficient and inefficient DMUs, for assessment and improving the capabilities of DEA. Khezrimotlagh *et al.* (2012) developed the "Arash method" which successfully differentiates between technical efficient and inefficient DMUs and rank them.

Differentiating between technical efficiency and efficiency is of great importance in the practical application of DEA. A little difference in efficiency evaluation can have a drastic impact on decision making for a decision maker. Therefore, sensitivity analysis on the efficiency frontier of DEA models is imperative. Technical efficient DMUs identified as efficient DMUs by previous DEA models such as the Anderson and Peterson (AP) super-efficiency model are debatable, because some technical efficient DMUs are more inefficient than some inefficient DMUs. Khezrimotlagh *et al.* (2012) rectifies the drawback by using a small error in input values of the data using Additive DEA model. Although this technique (AM) proves logical and practical, it does not take into consideration the weak part of the efficiency frontier or DMUs that take their efficiency score when compared to the weak part of the efficient frontier. We approach this drawback in this modification by placing an upper bound on the free variable of the multiplier side of Arash method.

The aim of this paper is to modify the Arash method and increase the discretionary power of the model. This is achieved using facet analysis. A similar modification was introduced by Daneshvar (2009) and Daneshvar *et al.* (2014) on Variable Return to Scale model (VRS). The rest of the paper is organized as follows. Section 2 discusses the

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materials and methods with a numerical illustration, and section 3 presents results and discussion. The paper is concluded in section 4.

2. Materials and methods

2.1. The Arash method

The Arash method was developed by Khezrimotlagh *et al.* (2012) to examine the Farrell frontier, and evaluate DMUs that do the job right and remove the drawbacks of arranging DMUs Additive DEA model. They achieved that by introducing a small error into the inputs of the observed DMUs. The Arash method identified the inadequacy in the definition of efficiency of Pareto and illustrated the limitations in DEA technique to bench and rank DMUs. The Arash method is capable of differentiating between technical efficient and/or inefficient DMUs without additional information in the form of weight restriction or statistical technique and super-efficiency, it also point out that, technical efficiency is a necessary condition for becoming efficient but it is not enough to call it efficient. In the absence of cost information, the Arash method is also capable of measuring the cost efficiency of DMUs.

To illustrate the method: Assume there are n DMUs $DMU_j(j = 1, \dots, n)$, with non-negative input $(x_{ij}, i = 1, \dots, m)$ and non-negative output $(y_{rj}, r = 1, \dots, s)$. For each DMU which has at least one of its inputs and one its outputs that is non-zero. The input-orientation case of $\varepsilon - AM$ is as follows. ε is the amount of error given to input. DMU_0 is evaluated DMU and $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$ $\varepsilon_j \geq 0$

$$Max \quad \sum_{i=1}^{m} w_i^- s_i^- + \sum_{r}^{s} w_r^+ s_r^+ \tag{1}$$

subject to

$$\sum_{j=1}^{n} \lambda x_{ij} + s_{j=1}^- = x_0 + \varepsilon_j \qquad i = 1, \dots, m \tag{2}$$

$$\sum_{j=1}^{n} \lambda y_{rj} - s_r^+ = y_0 \qquad r = 1, \dots, s \tag{3}$$

$$\sum_{j=1}^{n} \lambda = 1 \qquad j = 1, \dots, n \tag{4}$$

$$s_i^- \geq 0 \qquad i = 1, \dots, m \tag{5}$$

$$s_r^+ \geq 0 \qquad r = 1, \dots, s \tag{6}$$

The score of $\varepsilon - AM$ is marked by A_ε^* where $\varepsilon = \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$. The Arash method is designated by A_ε^* where $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ is the input error. Each technical efficient DMU under evaluation is compared with a technical efficient target suggested by the model, using a little difference in amount. Then, using the real definition of efficiency (Input/Output) it characterizes if the technical efficient DMU is efficient or not.

$$A_\varepsilon^* = \frac{p/m}{\sum_{k=1}^{p} (y_k^*/y_k) \Big/ \sum_{i=1}^{m} (x_j^*/x_j)} \tag{7}$$

If $A_{\varepsilon}^{*} < 1$ for an observed DMU, $\varepsilon - AM$ suggest that the observed DMU changes its input and output values to that of $\varepsilon - AM$ target, otherwise, if $A_{\varepsilon}^{*} \geq 1$, the $\varepsilon - AM$ suggest the DMU to remain the same, showing that it has a good combination of input and output values in the PPS, thus preventing it from decreasing its efficiency score.[1]

2.2. Facet analysis

In using facet analysis on the Arash method, we examine the intersection of production possibility set "T" and the plane $P = \{(X,Y) \mid X = \beta Y_0, Y = \beta Y_0, \alpha, \beta \geq 0\}$ it is illustrated as follows:

$$P \cap T = \left\{ (X, Y \mid X = \alpha X_O \geq \sum_{j=1}^{n} \lambda_j X_j, Y = \beta Y_O \leq \sum_{j=1}^{n} \lambda_j Y_j \right\} \quad \sum_{j=1}^{n} \lambda_j$$
$$\lambda_j \geq 0, \{\forall (j = 1, \ldots, n) \forall \alpha, \beta \geq 0\} \tag{8}$$

Banker and Thrall (1992) emphasized that the production possibility set may have more than one supporting hyperplanes at any efficient point. The values of $(u_0^{+} and\ u_0^{-})$ upper and lower bounds for all supporting hyperplanes that pass through $(\alpha\ and\ \beta)$ can be computed as follows:

$$Max\ u_0 \tag{9}$$
subject to
$$UY_0 + u_0 = 1 \tag{10}$$
$$UY_j - VX_j + u_0 \leq 0 \qquad j = 1, 2, \ldots, n \tag{11}$$
$$VX_0 = 1 \tag{12}$$
$$U \geq 0, \quad V \geq 0, \quad u_0\ free \tag{13}$$

$$Min\ u_0 \tag{14}$$
subject to
$$UY_0 + u_0 = 1 \tag{15}$$
$$UY_j - VX_j + u_0 \leq 0 \qquad j = 1, 2, \ldots, n \tag{16}$$
$$VX_0 = 1 \tag{17}$$
$$U \geq 0, \quad V \geq 0, \quad u_0\ free \tag{18}$$

$(u_0^{+} and\ u_0^{-})$ Denotes the optimal solution of model (9)-(13) and (14)-(18) respectively, u_0^{-} may resolve to (-∞) for some DMUs. The following inequality holds for classical BCC model$(u_0^{+} \leq u_0^{*} \leq u_0^{+})$.

The modification is based on the assumption that the technical efficient DMUs identified by the Arash method, are DMUs located at the weak part of the frontier or get their efficiency score when compared to those on the weak part of the frontier. Therefore, after the modification, the efficiency score of the efficient DMUs remain the same, only that

[1] **See** Khezrimotlagh et al. (2012). A new method in data envelopment analysis to find efficient decision making units and rank both technical efficient and inefficient DMUs together

of the technical efficient DMUs changes. Furthermore, the PPS of the Arash method is the same as the PPS of BCC model, because the primary model, which the Arash method is based on (Additive DEA model), has the same PPS as the BCC model. Therefore, reaffirming our assumption that similar modification achieved on the BCC model to get the modified VRS model by Daneshvar *et al.* (2014) which modifies the weak part of the frontier of the BCC model is possible on the Arash method.

3. Results and Discussion

3.1. The Proposed modification

We try to modify the PPS of the Arash method using facet analysis of Banker and Thrall by restricting the free variable u_0. The proposed model is named the modified Arash Method (MAM). To illustrate the proposed model, suppose there are n DMUs, $DMU_j (j = 1, \dots, n)$ with m non-negative input, x_{ij}, $(i = 1, \dots, m)$ and s non-negative output, y_{rj}, $(r = 1, \dots, s)$.

First compute the efficiency of the DMUs using the standard BCC model. Then apply the facet analysis model (14)-(18) to compute u_0^- for all efficient DMUs identified by BCC model. The upper bound for the proposed model is β, where $\beta = max\left[u_0^- \mid u_0^- \neq 1 \; for \; efficient \; DMUs\right]$.

The standard Arash method is modified by computing the dual of the Arash method model (19)-(25) and placing β as an upper bound for the free variable u_0 as follows:

$$Min \sum_{i=1}^{m} V_i \left(x_0 + \varepsilon_j^-\right) + \sum_{r=1}^{s} U_r y_0 + u_0 \quad (19)$$

subject to

$$\sum_{j=1}^{n} V_i x_{ij} + \sum_{j=1}^{n} U_r y_{ij} + u_0 \geq 0 \quad (20)$$

$$V_i \geq w_j^- \quad (21)$$

$$-Ur \geq w_k^+ \quad (22)$$

$$u_0 \leq \beta \quad (23)$$

$$V_i \;\; free \quad (24)$$

$$U_r \;\; free \quad (25)$$

The dual of model (19)-(25) illustrates the proposed modified Arash method (MAM)

$$Max \;\; \sum_{i=1}^{m} w_i^- s_i^- + \sum_{r=1}^{s} w_r^+ s_r^+ + \eta\beta \quad (26)$$

subject to

$$\sum_{i=1}^{n} \lambda x_{ij} + s_i^- = x_0 + \varepsilon_j^- \quad (27)$$

$$\sum_{i=1}^{n} \lambda y_{ij} - s_r^+ = y_0 \quad (28)$$

$$\sum_{i=1}^{n} \lambda_j + \eta = 1 \quad (29)$$

$$\lambda \geq 0 \qquad j = 1, \dots, n \quad (30)$$

$$s_i^- \geq 0 \qquad i = 1, \ldots, m \qquad (31)$$

$$s_r^+ \geq 0 \qquad r = 1, \ldots s \qquad (32)$$

$$\eta \geq 0 \qquad (33)$$

The weights for the model w_i^- *and* w_r^+ are defined as follows:

$$\begin{bmatrix} \varepsilon_j^- = \varepsilon \times \min(x_i, y_r) \\ w_i^- = \begin{cases} \frac{1}{x_{ij}} & x_{ij} \neq 0 \\ N_i & x_{ij} = 0 \end{cases} \\ w_r^+ = \begin{cases} \frac{1}{y_{rj}} & y_{rj} \neq 0 \\ M_r & y_{rj} = 0 \end{cases} \end{bmatrix} \begin{pmatrix} N_i \text{ and } M_r \text{ can be selected from} \\ \text{a positive real number set depending} \\ \text{on the goals of the DMUs resources} \\ \text{and production} \end{pmatrix} \qquad (34)$$

$$Targets: x_{ij}^* = \begin{cases} x_{ij} + \varepsilon_j^- - s_i^*, \forall i, \\ y_{rj}^* = y_{rj} + s_r^*, \forall r, \end{cases}$$

$$Score: \quad A^* = \frac{\sum_{r=1}^{s} w_r^+ y_{rj} \Big/ \sum_{i=1}^{m} w_i^- x_{ij}}{\sum_{r=1}^{s} w_r^+ y_0^* \Big/ \sum_{i=1}^{m} w_i^- x_0^*} \qquad (35)$$

Model (26)-(33) does not change the efficiency value of efficient and strong efficient DMUs, changes are only in the efficiency value of weak efficient (technical efficient) DMUs and DMUs that are compared with weak frontier.

3.2. Numerical example

Table 1 shows the data set with their corresponding efficiency scores and u_0^- values. This is the same set of data used by Khezrimotlagh *et al.* (2014) in DEA2013 conference Samsun Turkey. We added some DMUs to place them at the weak part of the frontier to better illustrate the effectiveness of the proposed model. From Table 1, DMUs A, B, G, H and I are BCC efficient, therefore we computed u_0^- for the efficient DMUs to get the upper bound for the free variable of the proposed MAM. From Table the fourth column, the β value is 0.8571 for the set of evaluated DMUs.Table 2 and Table 3 summarizes the results and ranking of the evaluated DMUs. WinqQSB 2.0 linear programming software was used for the analysis.

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Table 1. Input/output data with efficiency and facet analysis.

DMUs	Input	Output	BCC	u_o^-
A	2	2	1	0.8571
B	3	9	1	-20
C	10	10	0.98	*****
D	3	8.7	0.967	*****
E	3.3	9	0.9091	*****
F	10.3	10	0.9514	*****
G	9.8	10	1	$-\infty$
H	2	1	1	1
I	2	1.5	1	1

Table 2. The results of ε-AM and ε-MAM.

DMUs	0-AM	Rank	0-MAM	Rank	0.1-AM	Rank	0.1-MAM	Rank
A	1	1	-1.238	7	0.65	7	-1.405	7
B	1	1	1	1	0.9667	3	0.9667	3
C	0.98	4	0.98	3	0.97	2	0.97	2
D	0.9656	5	0.9383	5	0.9323	5	0.905	5
E	0.9091	7	0.9091	6	0.8788	6	0.8788	6
F	0.9515	6	0.9515	4	0.9417	4	0.9417	4
G	1	1	1	1	0.9898	1	0.9898	1
H	0	9	-4.2833	9	-0.7	9	-4.555	9
I	0.6667	8	-2.283	8	0.2008	8	-2.4547	8

Table 3. The results of 0.5-AM and 0.5-MAM.

DMUs	0.5-AM	Rank	0.5-MAM	Rank
A	-0.75	7	-1.8917	7
B	0.8334	4	0.8334	4
C	0.93	2	0.93	2
D	0.7989	6	0.7717	6
E	0.7576	5	0.7576	5
F	0.9029	3	0.9029	3
G	0.949	1	0.949	1
H	-3.5	9	-5.6417	9
I	-1.664	8	-3.1413	8

From Table 2 column 2, 0-AM showed three efficient DMUs and six inefficient DMUs, 0-MAM, which immediately disagrees with the values of 0-AM, suggesting that DMU A is more inefficient than the inefficient DMUs C, D E and F. The 0-AM ranked DMU A among the first while 0-MAM ranks it as seven. This is clearly logical because, if

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DMU B uses three units of inputs to produce nine outputs, then DMU A is very inefficient for producing only two outputs with two inputs. 0.1-AM shows that, DMU A is more inefficient than the inefficient DMUs C, D, E, F and G. (0.1-MAM) shows that DMU A is worse than it appears, by reducing its value. DMUs A, D, H and I in Table 2 and Table 3 show the changes in the values of the DMUs. Suggesting that they are at the weak part of the frontier or are compared to DMUs located at the weak part of the frontier. Other unchanged values are located at the strong part of the frontier or are compared to those located at the strong part of the frontier.

The managerial and technological application of this model is clear, assigning an exaggerated value to a business or unit has a significant financial implication. The proposed model does not exaggerate the true value of efficient DMUs.

4. Conclusion

In this paper, we applied facet analysis on the Arash method by using an upper bound on the free variable of the multiplier side. The results of Table 2 and Table 3 clearly shows that the Arash method gives an exaggerated efficiency value to DMUs located at the weak part of the efficiency frontier or DMUs compared to the weak part of the frontier, otherwise known as technically efficient and weak efficient DMUs. By placing an upper bound on the free variable of the Arash method, we fix this slight drawback by extending the stability region of the efficiency frontier. The values of the MAM are justified and clearly effective, because, it shows that a little difference in input or output is significant in identifying the entities that do the job right and those that can improve their performance.

The proposed modal can be considered as a pessimistic modal, because it focuses on the weak part of the efficiency frontier. A pessimistic point of view may be the view of all managers and decision makers, because the risk of losing finance and resources is minimized. Therefore, the proposed MAM shows the true performance of a DMU as suppose to the overly exaggerated efficiency score.

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