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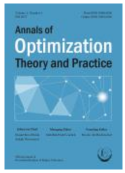

Policy decision making based on some averaging aggregation operators of t-spherical fuzzy sets; a multi-attribute decision making approach

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Abstract Multi-attribute decision making (MADM) is a hot research area in fuzzy mathematics and to deal with that, the averaging and geometric aggregation operators (AOs) are the widely used tools. The aim of this manuscript is to propose the notion of averaging and geometric AOs in the environment of T-spherical fuzzy sets (TSFSs). TSFS enables the selection of grades of memberships from considerably a larger domain and hence overcome the drawbacks of the existing fuzzy frameworks. In this paper, we develop some novel operations for TSFSs including algebraic sum, product etc. Based on new operations some averaging AOs including T-spherical fuzzy weighted averaging (TSFWA) and T-spherical fuzzy weighted geometric (TSFWG) operators are developed. The monotonicity, idempotency and boundedness of the defined operators are investigated, and their fitness is validated using induction method. With the help of an illustrative example, the problem of policy decision making using a MADM algorithm is solved. The new proposed work and the existing literature is compared numerically and the advantages of the TSFWA and TSFWG operators are investigated over existing work.

Keywords Aggregation operators; T-spherical fuzzy set; Picture fuzzy set; Multi-attribute decision making; Spherical fuzzy set

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Policy Decision Making Based on Some Averaging Aggregation Operators of T-Spherical Fuzzy Sets; A Multi-Attribute Decision Making Approach

Abstract: Multi-attribute decision making (MADM) is a hot research area in fuzzy mathematics and to deal with that, the averaging and geometric aggregation operators (AOs) are the widely used tools. The aim of this manuscript is to propose the notion of averaging and geometric AOs in the environment of T-spherical fuzzy sets (TSFSs). TSFS enables the selection of grades of memberships from considerably a larger domain and hence overcome the drawbacks of the existing fuzzy frameworks. In this paper, we develop some novel operations for TSFSs including algebraic sum, product etc. Based on new operations some averaging AOs including T-spherical fuzzy weighted averaging (TSFWA) and T-spherical fuzzy weighted geometric (TSFWG) operators are developed. The monotonicity, idempotency and boundedness of the defined operators are investigated, and their fitness is validated using induction method. With the help of an illustrative example, the problem of policy decision making using a MADM algorithm is solved. The new proposed work and the existing literature is compared numerically and the advantages of the TSFWA and TSFWG operators are investigated over existing work.

Keywords: Aggregation Operators; T-spherical fuzzy set; Picture fuzzy set; Multi-attribute decision making; spherical fuzzy set.

1. Introduction:

After Zadeh's [1] remarkable work on fuzzy set (FS), scientists took interest in theory of FS as a FS can deal with real life scenarios with more accuracy and precision. Zadeh's FS leaded the researchers in many new directions such as interval-valued FS [2], IFS and intuitionistic fuzzy number (IFN) [3, 4], interval-valued IFS [5], hesitant fuzzy set (HFS) [6, 7] and Pythagorean FS (PyFS) [8, 9] etc. These new structures are developed due to the deficiency in the existing literature and can deal with real-life problems with more accuracy. Atanassov's IFS improved Zadeh's FS as it introduced the non-membership grade along with the membership grade. With two membership functions, Atanassov's IFSs modeled real-life scenarios effectively but there are some situations where it fails to be applied. In such scenarios the framework of PFS and picture fuzzy number (PFN), presented by Cuong [10] that uses four membership functions namely membership, abstinence, non-membership and refusal degree, could be a better option. For a better understanding, Consider the following two real-life scenarios:

1. In electoral voting, a voter's behavior could be divided into four types i.e. vote in favor, vote against, abstinence or refusal. Here, the abstain mean that one may vote in favor and against at the same time i.e. one can leave the ballet paper blank or stamped in favor and against both but in any way casted the vote. The refusal degree is based on those voters who did not vote at all.
2. In MADM problems, whenever a candidate is evaluated, the information about him could be true $+ve$, true $-ve$, false $-ve$ and false $+ve$ which could be modeled by four characteristic functions of PFS.

The above two examples clearly proved the generalization of PFS over IFS. Due to the diverse nature of PFS, some quality results in this direction have been achieved such as in [11] the authors introduced the idea of PFSs which proves to be a generalized form of IFSs. [12] is about some fuzzy logic operators for PFSs, in reference [13] Garg studied some aggregation operators (AOs) in picture fuzzy settings. [14] investigated picture fuzzy cross-entropy for MADM problems, and reference [15] developed some picture fuzzy AOs and investigated their practicality in decision making problems. For other related work see [16, 17].

As discussed, Cuong's idea of PFS is substantial due to its flexible nature that the existing concepts lacks but there is a drawback in it i.e. the sum of grades of all four functions in a PFS must be less than one. This is a major drawback in the structure of PFS as due to this factor one is unable to assign values to membership functions independently. Realizing this issue, the concept of spherical fuzzy set (SFS) and spherical fuzzy number (SFN) were introduced by Mahmood et. al. [18] which were further extended to TSFS and T-spherical fuzzy number (TSFN) in 2018. The idea of TSFS is a generalization of IFS and PFS allowing the grades of membership functions to be chosen from anywhere in the unit interval without any restriction (See Definition 4 and 5). TSFSs have been successfully applied to some real problems as in [18], the framework of TSFS is used in medical diagnosis and MADM problems. In [19], some similarity measures are defined for TSFSs and applied in pattern recognition problems. The framework of TSFS is so diverse and flexible that it can solve all the problems lies in the environment of IFS and PFS conveniently however, IFS and PFS on the other hand could not be able to deal with the information provided in the environment of TSFSs. Further, Ullah et al. [20] evaluated the investment policy making based interval-valued T-Spherical fuzzy aggregation using a MADM approach. Ullah et al. [21] investigated the cluster analysis using correlation coefficients for TSFSs. For some other recent work one may refer to [22, 23].

MADM is a process in which individuals or objects are evaluated based on some attributes with the help of some aggregation tools. Since the theory of FS has been developed, the concept of FS and its several extensions have been applied to MADM problems e.g. in [24] FSs are used in medical diagnosis, [25] is about decision making in fuzzy environment. In MADM problems, aggregation tools play essential role. Based on t-norms and t-conorms, several AOs have been developed so far including averaging, geometric, Einstein averaging and Archimedean averaging operators have been developed for several different extensions of FSs such as, Xu [26] studied MADM problems using similarity measures of IFSs, [27] introduced a novel approach to MADM based on IFSs, in [28] Liu et al. studied MADM methods based on IFSs, Reference [29] focused on MADM method using on a new accuracy formula under interval-valued IFSs and in Reference [30] Xu investigated the role of power AOs in MADM problems under intuitionistic fuzzy settings. Some other related work on theory of AOs and MADM in fuzzy environment one may refer to [31-52].

In [31] Xu developed some averaging AOs of IFSs and applied them in MADM problems. [9] developed the theory of averaging operators for Pythagorean FSs as a generalization of averaging operators of IFSs. Garg [13] extended the idea of Xu [31] and developed the same operators in the environment of PFSs. Garg proved that the operators defined in [31] could not be applied in situations discussed in [13] i.e. in the environment of PFSs and that the operators defined in [13] can process the information provided in the framework of IFSs and PFSs as well.

In this article, we aim to develop the averaging and geometric AOs of TSFSs as a generalization of operators defined in [9, 13, 31]. The properties of the new AOs are discussed along with pointing out the shortcomings of the existing AOs and the advantages of the proposed AOs.

The article is organized as: section one provided a brief introduction of existing literature, pointing out towards possible drawbacks of existing concepts and mentioning some spaces for new work. Section two is based on some prerequisites. In section three, some new set theoretic operations for TSFSs are proposed and their results are studied. In section four, based on defined operations, some new averaging AOs are defined for TSFSs, their properties are investigated and their generalizations over the existing literature is proved. In section five, the theory of geometric AOs is discussed. In section six, an algorithm for MADM process in T-spherical fuzzy environment is developed and a numerical example is solved using proposed algorithm. Section seven involves the comparison of the new and existing literature and proposed some advantages of new work. In section eight, some concluding remarks are added summarizing the article and some future study is discussed.

2. Preliminaries:

In preliminaries, some pre-requisites are illustrated in support of proposed new work. For a better understanding regarding this work, please refer to [3, 11, 13, 18, 31]. In our study ahead, we use the four functions where s will represent membership grade, i denote the abstinence grade, d standes for non-membership grade and r represents refusal degree of an element.

Atanassov [3] developed the notion of IFS by considering the dissatisfaction (non-membership) degree of an uncertain event as well along with the satisfaction (membership) degree. The notion of IFS also described the hesitancy degree of an event in view of membership and non-membership degree.

Definition 1: [3] On a set X, an IFS is of the shape $I = \{(s(x), d(x)): 0 \leq sum(s, d) \leq 1\}$. Further, $r(x) = 1 - sum(s, d)$ represents the hesitancy level of $x \in X$ and (s, d) is termed as an IFN.

To improve the restriction i.e. $0 \leq sum(s, d) \leq 1$ subjected on Atanassov's IFS, Yager [8] developed the notion of PyFS which provides a larger range of space to choose the values of the membership and non-membership degrees from unit interval.

Definition 2: [8] On a set X, a PyFS is of the shape $P = \{(s(x), d(x)): 0 \leq sum(s^2, d^2) \leq 1\}$. Further, $r(x) = \sqrt{1 - sum(s^2, d^2)}$ represents the hesitancy level of $x \in X$ and (s, d) is termed as a PyFN.

Cuong [10] was the one who feels that IFS and PyFS does not cover all the aspects of human opinion. He suggested that a human opinion not only involve satisfaction and dissatisfaction degree but in has some sort of abstinence and refusal component as well. To count all these factors of human opinion, Cuong [10] proposed a new framework of PFS which is described as follows.

Definition 3: [10] On a set X, a PFS is of the shape $P = \{(s(x), i(x), d(x)): 0 \leq sum(s, i, d) \leq 1\}$. Further, $r(x) = 1 - sum(s, i, d)$ represents the refusal degree of $x \in X$ and (s, i, d) is termed as a PFN.

Following the pattern of Yager, Mahmood et al. [18] sensed the limitation of PFS i.e. $0 \leq sum(s, i, d) \leq 1$. This restriction makes the framework of PFS a limited one and do not allow us to allot values to all three grades a value from $[0, 1]$ interval. To overcome this issue, Mahmood et al [18] introduced the notion of SFS and TSFS. Both these concepts are defined as follows.

Definition 4: [18] On a set X, a SFS is of the shape $S = \{(s(x), i(x), d(x)): 0 \leq sum(s^2, i^2, d^2) \leq 1\}$. Further, $r(x) = \sqrt{1 - sum(s^2, i^2, d^2)}$ represents the refusal degree of $x \in X$ and (s, i, d) is termed as a SFN.

Definition 5: [18] On a set X, a TSFS is of the shape $\mathcal{T} = \{(s(x), i(x), d(x)): 0 \leq sum(s^n, i^n, d^n) \leq 1\}$ for $n \in \mathbb{Z}^+$. Further, $r(x) = \sqrt[n]{1 - sum(s^n, i^n, d^n)}$ represents the refusal degree of $x \in X$ and (s, i, d) is termed as a TSFN.

The remark below shows the generalization of TSFS over existing approaches. Some restrictions are stated under which a TSFS reduces to existing literature.

Remark 1: [18] The TSFS reduces to:

1. SFS for $n = 2$. (See Definition 4)
2. PFS for $n = 1$. (See Definition 3)
3. PyFS for $n = 2$ and $i = 0$. (See Definition 2)
4. IFS for $n = 1$ and $i = 0$. (See Definition 1)

The set theoretic operations of TSFSs including union, inclusion, intersection and complement were defined by Mahmood et al [18] and are described as:

Definition 6: [18] For two TSFNs $\mathcal{A}$ and β, their basic operations are:

1. $\mathcal{A} \subseteq \beta$ iff $s_{\mathcal{A}} \preccurlyeq s_{\beta}, i_{\mathcal{A}} \preccurlyeq i_{\beta}, d_{\mathcal{A}} \succcurlyeq d_{\beta}$ for all $x \in X$.
2. $\mathcal{A} = \beta$ iff $\mathcal{A} \subseteq \beta$ and $\beta \subseteq \mathcal{A}$.
3. $\mathcal{A} \cup \beta = \left\{\langle x, \curlyvee\left(s_{\mathcal{A}}(x), s_{\beta}(x)\right), \curlywedge\left(i_{\mathcal{A}}(x), i_{\beta}(x)\right), \curlywedge\left(d_{\mathcal{A}}(x), d_{\mathcal{A}}(x)\right)\rangle : x \in X\right\}$.
4. $\mathcal{A} \cap \beta = \left\{\langle x, \curlywedge\left(s_{\mathcal{A}}(x), s_{\beta}(x)\right), \curlywedge\left(i_{\mathcal{A}}(x), i_{\beta}(x)\right), \curlyvee\left(d_{\mathcal{A}}(x), d_{\mathcal{A}}(x)\right)\rangle : x \in X\right\}$.
5. $\mathcal{A}^c = \{\langle x, d_{\mathcal{A}}(x), i_{\mathcal{A}}(x), s_{\mathcal{A}}(x)\rangle : x \in X\}$.

As the manuscript is based on some new AOs and their applications in MADM problems where comparison of data is essential. Therefore, we need some rules for comparing TSFNs. Here we list the comparison rules defined by Mahmood et al. [18] which are the generalizations of comparison rules of IFSs [31] and PFSs [13, 15].

Definition 7: [18] Let $\mathcal{A}$ be a TSFN for some $n \in \mathbb{Z}^+$. The score value Š and accuracy value Â of $\mathcal{A}$ are of the form:

$$\check{S}(\mathcal{A}) = s_{\mathcal{A}}^n(x) - d_{\mathcal{A}}^n(x), \check{S}(\mathcal{A}) \in [-1, 1]$$

$$\hat{A}(\mathcal{A}) = s_{\mathcal{A}}^n(x) + i_{\mathcal{A}}^n(x) + d_{\mathcal{A}}^n(x), \hat{A}(\mathcal{A}) \in [0, 1].$$

If the scores of two TSFNs are distinct. Then

- $\mathcal{A} > \beta$ if $\check{S}(\mathcal{A}) > \check{S}(\beta)$.
- $\mathcal{A} < \beta$ if $\check{S}(\mathcal{A}) < \check{S}(\beta)$.

Whenever the scores of two TSFNs becomes indifferent i.e. $\check{S}(\mathcal{A}) = \check{S}(\beta)$. Then

- $\mathcal{A} > \beta$ if $\hat{A}(\mathcal{A}) > \hat{A}(\beta)$.
- $\mathcal{A} < \beta$ if $\hat{A}(\mathcal{A}) < \hat{A}(\beta)$.
- $\mathcal{A} \approx \beta$ if $\hat{A}(\mathcal{A}) = \hat{A}(\beta)$.

3. Some New Operations for TSFSs

Our aim in this section is to develope some new operations for TSFSs and investigated their results. These operations are the generalizations of operations of IFSs, Pythagorean FSs and PFSs providing a base for the AOs developed in next section. Moreover, the proposed operations are the generalizations of the operations developed in [9, 11, 13, 31]

Definition 8: Consider two TSFNs $\mathcal{A}$ and β for some $n \in \mathbb{Z}^+$ and let $\lambda > 0$. Then:

1. $\mathcal{A} \oplus \beta = \left(\sqrt[n]{s_{\mathcal{A}}^n + s_{\beta}^n - s_{\mathcal{A}}^n . s_{\beta}^n}, i_{\mathcal{A}} . i_{\beta}, d_{\mathcal{A}} . d_{\beta} \right)$.
2. $\mathcal{A} \otimes \beta = \left(s_{\mathcal{A}} . s_{\beta}, i_{\mathcal{A}} . i_{\beta}, \sqrt[n]{d_{\mathcal{A}}^n + d_{\beta}^n - d_{\mathcal{A}}^n . d_{\beta}^n} \right)$
3. $\lambda . \mathcal{A} = \left(\sqrt[n]{1 - (1 - s_{\mathcal{A}}^n)^{\lambda}}, (i_{\mathcal{A}})^{\lambda}, (d_{\mathcal{A}})^{\lambda} \right)$.
4. $\mathcal{A}^{\lambda} = \left((s_{\mathcal{A}})^{\lambda}, (i_{\mathcal{A}})^{\lambda}, \sqrt[n]{1 - (1 - d_{\mathcal{A}}^n)^{\lambda}} \right)$

Remark 2: Obviously $\mathcal{A} \oplus \beta$, $\mathcal{A} \otimes \beta$, $\lambda . \mathcal{A}$ and $\mathcal{A}^{\lambda}$ are TSFNs.

Remark 3: The operations proposed in Definition 8 reduces to spherical fuzzy environment for $n = 2$ and are given by:

1. $\mathcal{A} \oplus \beta = \left(\sqrt{s_{\mathcal{A}}^2 + s_{\beta}^2 - s_{\mathcal{A}}^2 . s_{\beta}^2}, i_{\mathcal{A}} . i_{\beta}, d_{\mathcal{A}} . d_{\beta} \right)$.
2. $\mathcal{A} \otimes \beta = \left(s_{\mathcal{A}} . s_{\beta}, i_{\mathcal{A}} . i_{\beta}, \sqrt{d_{\mathcal{A}}^2 + d_{\beta}^2 - d_{\mathcal{A}}^2 . d_{\beta}^2} \right)$
3. $\lambda . \mathcal{A} = \left(\sqrt{1 - (1 - s_{\mathcal{A}}^2)^{\lambda}}, (i_{\mathcal{A}})^{\lambda}, (d_{\mathcal{A}})^{\lambda} \right)$.
4. $\mathcal{A}^{\lambda} = \left((s_{\mathcal{A}})^{\lambda}, (i_{\mathcal{A}})^{\lambda}, \sqrt{1 - (1 - d_{\mathcal{A}}^2)^{\lambda}} \right)$.

Remark 4: The operations proposed in Definition 8 reduces to picture fuzzy settings for $n = 1$ and are given by:

1. $\mathcal{A} \oplus \beta = \left(s_{\mathcal{A}} + s_{\beta} - s_{\mathcal{A}} . s_{\beta}, i_{\mathcal{A}} . i_{\beta}, d_{\mathcal{A}} . d_{\beta} \right)$.
2. $\mathcal{A} \otimes \beta = \left(s_{\mathcal{A}} . s_{\beta}, i_{\mathcal{A}} . i_{\beta}, d_{\mathcal{A}} + d_{\beta} - d_{\mathcal{A}} . d_{\beta} \right)$
3. $\lambda . \mathcal{A} = \left(1 - (1 - s_{\mathcal{A}})^{\lambda}, (i_{\mathcal{A}})^{\lambda}, (d_{\mathcal{A}})^{\lambda} \right)$.
4. $\mathcal{A}^{\lambda} = \left((s_{\mathcal{A}})^{\lambda}, (i_{\mathcal{A}})^{\lambda}, 1 - (1 - d_{\mathcal{A}})^{\lambda} \right)$.

Remark 5: The operations proposed in Definition 8 reduces to Pythagorean fuzzy environment for $n = 2$ and $i_{\mathcal{A}} = i_{\beta} = 0$ and are given by:

1. $\mathcal{A} \oplus \beta = \left(\sqrt{s_{\mathcal{A}}^2 + s_{\beta}^2 - s_{\mathcal{A}}^2 . s_{\beta}^2}, d_{\mathcal{A}} . d_{\beta} \right)$.
2. $\mathcal{A} \otimes \beta = \left(s_{\mathcal{A}} . s_{\beta}, \sqrt{d_{\mathcal{A}}^2 + d_{\beta}^2 - d_{\mathcal{A}}^2 . d_{\beta}^2} \right)$
3. $\lambda . \mathcal{A} = \left(\sqrt{1 - (1 - s_{\mathcal{A}}^2)^{\lambda}}, (d_{\mathcal{A}})^{\lambda} \right)$.
4. $\mathcal{A}^{\lambda} = \left((s_{\mathcal{A}})^{\lambda}, \sqrt{1 - (1 - d_{\mathcal{A}}^2)^{\lambda}} \right)$.

Remark 6: The operations proposed in Definition 8 reduces to intuitionistic fuzzy settings for $n = 1$ and $i_{\mathcal{A}} = i_{\beta} = 0$ are given by:

1. $\mathcal{A} \oplus \beta = \left(s_{\mathcal{A}} + s_\beta - s_{\mathcal{A}}.s_\beta, d_{\mathcal{A}}.d_\beta\right)$.
2. $\mathcal{A} \otimes \beta = \left(s_{\mathcal{A}}.s_\beta, d_{\mathcal{A}} + d_\beta - d_{\mathcal{A}}.d_\beta\right)$
3. $\lambda.\mathcal{A} = \left(1 - (1 - s_{\mathcal{A}})^\lambda, (d_{\mathcal{A}})^\lambda\right)$.
4. $\mathcal{A}^\lambda = \left((s_{\mathcal{A}})^\lambda, 1 - (1 - d_{\mathcal{A}})^\lambda\right)$.

Theorem 1: For two TSFNs $\mathcal{A}$ and β and for $\lambda, \lambda_1, \lambda_2 > 0$, the following holds:

1. $\mathcal{A} \oplus \beta = \beta \oplus \mathcal{A}$
2. $\mathcal{A} \otimes \beta = \beta \otimes \mathcal{A}$
3. $\lambda(\mathcal{A} \oplus \beta) = \lambda\mathcal{A} \oplus \lambda\beta$
4. $(\mathcal{A} \otimes \beta)^\lambda = \mathcal{A}^\lambda \otimes \beta^\lambda$
5. $\lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} = (\lambda_1 + \lambda_2)\mathcal{A}$
6. $\mathcal{A}^{\lambda_1} \otimes \mathcal{A}^{\lambda_2} = \mathcal{A}^{\lambda_1 + \lambda_2}$
7. $(\mathcal{A}^c)^\lambda = (\lambda\mathcal{A})^c$
8. $\lambda(\mathcal{A}^c) = \left(\mathcal{A}^\lambda\right)^c$
9. $\mathcal{A}^c \oplus \beta^c = (\mathcal{A} \otimes \beta)^c$
10. $\mathcal{A}^c \otimes \beta^c = (\mathcal{A} \oplus \beta)^c$

Proof: The proof for result 1, 3, 5, 7 and 9 are provided below. The remaining results could be proved analogously. Let $\mathcal{A} = (s_{\mathcal{A}}, i_{\mathcal{A}}, d_{\mathcal{A}})$ and $\beta = \left(s_\beta, i_\beta, d_\beta\right)$ and $\lambda, \lambda_1, \lambda_2 > 0$. Then

1. $\mathcal{A} \oplus \beta = \left(\sqrt[n]{s_{\mathcal{A}}^n + s_\beta^n - s_{\mathcal{A}}^n.s_\beta^n}, i_{\mathcal{A}}.i_\beta, d_{\mathcal{A}}.d_\beta\right)$

$$= \left(\sqrt[n]{s_\beta^n + s_{\mathcal{A}}^n - s_\beta^n.s_{\mathcal{A}}^n}, i_\beta.i_{\mathcal{A}}, d_\beta.d_{\mathcal{A}}\right)$$

$$= \beta \oplus \mathcal{A}$$

3. $\lambda(\mathcal{A} \oplus \beta) = \lambda\left(\sqrt[n]{s_{\mathcal{A}}^n + s_\beta^n - s_{\mathcal{A}}^n.s_\beta^n}, i_{\mathcal{A}}.i_\beta, d_{\mathcal{A}}.d_\beta\right)$

$$= \left(\sqrt[n]{1 - \left(1 - \left(\sqrt[n]{s_{\mathcal{A}}^n + s_\beta^n - s_{\mathcal{A}}^n.s_\beta^n}\right)^n\right)^\lambda}, \left(i_{\mathcal{A}}.i_\beta\right)^\lambda, \left(d_{\mathcal{A}}.d_\beta\right)^\lambda\right)$$

$$= \left(\sqrt[n]{1 - \left(1 - \left(s_{\mathcal{A}}^n + s_\beta^n - s_{\mathcal{A}}^n.s_\beta^n\right)\right)^\lambda}, (i_{\mathcal{A}})^\lambda\left(i_\beta\right)^\lambda, (d_{\mathcal{A}})^\lambda\left(d_\beta\right)^\lambda\right) \quad ...(1')$$

$\lambda\mathcal{A} \oplus \lambda\beta = \lambda(s_{\mathcal{A}}, i_{\mathcal{A}}, d_{\mathcal{A}}) \oplus \lambda\left(s_\beta, i_\beta, d_\beta\right)$

$$= \left(\sqrt[n]{1 - (1 - s_{\mathcal{A}}^n)^\lambda}, i_{\mathcal{A}}, d_{\mathcal{A}}\right) \oplus \left(\sqrt[n]{1 - \left(1 - s_\beta^n\right)^\lambda}, i_\beta, d_\beta\right)$$

$$= \left(\sqrt[n]{\left(\sqrt[n]{1-(1-s_{\mathcal{A}}^n)^\lambda}\right)^n + \left(\sqrt[n]{1-\left(1-s_\beta^n\right)^\lambda}\right)^n - \left(\sqrt[n]{1-(1-s_{\mathcal{A}}^n)^\lambda}\right)^n . \left(\sqrt[n]{1-\left(1-s_\beta^n\right)^\lambda}\right)^n}, \right.$$

$$\left. (i_{\mathcal{A}})^\lambda (i_\beta)^\lambda, (d_{\mathcal{A}})^\lambda (d_\beta)^\lambda \right)$$

$$= \left(\sqrt[n]{(1-(1-s_{\mathcal{A}}^n)^\lambda) + \left(1-\left(1-s_\beta^n\right)^\lambda\right) - (1-(1-s_{\mathcal{A}}^n)^\lambda).\left(1-\left(1-s_\beta^n\right)^\lambda\right)}, (i_{\mathcal{A}})^\lambda (i_\beta)^\lambda, (d_{\mathcal{A}})^\lambda (d_\beta)^\lambda \right)$$

$$= \left(\sqrt[n]{1-(1-s_{\mathcal{A}}^n)^\lambda . \left(1-s_\beta^n\right)^\lambda}, (i_{\mathcal{A}})^\lambda (i_\beta)^\lambda, (d_{\mathcal{A}})^\lambda (d_\beta)^\lambda \right)$$

$$= \left(\sqrt[n]{1-\left(1-\left(s_{\mathcal{A}}^n + s_\beta^n - s_{\mathcal{A}}^n . s_\beta^n\right)\right)^\lambda}, (i_{\mathcal{A}})^\lambda (i_\beta)^\lambda, (d_{\mathcal{A}})^\lambda (d_\beta)^\lambda \right) \quad \ldots (2')$$

Using $(1')$ and $(2')$, we have $\lambda(\mathcal{A} \oplus \beta) = \lambda\mathcal{A} \oplus \lambda\beta$

5. $\lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} = \left(\sqrt[n]{1-(1-s_{\mathcal{A}}^n)^{\lambda_1}}, (i_{\mathcal{A}})^{\lambda_1}, (d_{\mathcal{A}})^{\lambda_1} \right) \oplus \left(\sqrt[n]{1-(1-s_{\mathcal{A}}^n)^{\lambda_2}}, (i_{\mathcal{A}})^{\lambda_2}, (d_{\mathcal{A}})^{\lambda_2} \right)$

$$= \left(\sqrt[n]{\left(\sqrt[n]{1-(1-s_{\mathcal{A}}^n)^{\lambda_1}}\right)^n + \left(\sqrt[n]{1-(1-s_{\mathcal{A}}^n)^{\lambda_2}}\right)^n - \left(\sqrt[n]{1-(1-s_{\mathcal{A}}^n)^{\lambda_1}}\right)^n . \left(\sqrt[n]{1-(1-s_{\mathcal{A}}^n)^{\lambda_2}}\right)^n}, \right.$$

$$\left. (i_{\mathcal{A}})^{\lambda_1} (i_{\mathcal{A}})^{\lambda_2}, (d_{\mathcal{A}})^{\lambda_1} (d_{\mathcal{A}})^{\lambda_2} \right)$$

Proceeding as we did in Proof of 3, we have,

$$= \left(\sqrt[n]{1-(1-s_{\mathcal{A}}^n)^\lambda . (1-s_{\mathcal{A}}^n)^\lambda},\ (i_{\mathcal{A}})^{\lambda_1} (i_{\mathcal{A}})^{\lambda_2}, (d_{\mathcal{A}})^{\lambda_1} (d_{\mathcal{A}})^{\lambda_2} \right)$$

$$= \left(\sqrt[n]{1-(1-s_{\mathcal{A}}^n)^{\lambda_1} . (1-s_{\mathcal{A}}^n)^{\lambda_2}}, (i_{\mathcal{A}})^{\lambda_1} (i_{\mathcal{A}})^{\lambda_2}, (d_{\mathcal{A}})^{\lambda_1} (d_{\mathcal{A}})^{\lambda_2} \right)$$

$$= \left(\sqrt[n]{1-(1-s_{\mathcal{A}}^n)^{\lambda_1+\lambda_2}}, (i_{\mathcal{A}})^{\lambda_1+\lambda_2}, (d_{\mathcal{A}})^{\lambda_1+\lambda_2} \right)$$

$$= (\lambda_1 + \lambda_2)\mathcal{A}$$

7. $(\mathcal{A}^c)^\lambda = ((s_{\mathcal{A}}, i_{\mathcal{A}}, d_{\mathcal{A}})^c)^\lambda = (d_{\mathcal{A}}, i_{\mathcal{A}}, s_{\mathcal{A}})^\lambda$

$$= \left((d_{\mathcal{A}})^\lambda, (i_{\mathcal{A}})^\lambda, \sqrt[n]{1-(1-s_{\mathcal{A}}^n)^\lambda} \right)$$

$$= (\lambda\mathcal{A})^c$$

9. $\mathcal{A}^c \oplus \beta^c = (d_{\mathcal{A}}, i_{\mathcal{A}}, s_{\mathcal{A}}) \oplus \left(d_\beta, i_\beta, s_\beta\right)$

$$= \left(\sqrt[n]{d_{\mathcal{A}}^n + d_\beta^n - d_{\mathcal{A}}^n . d_\beta^n}, i_{\mathcal{A}} . i_\beta, s_{\mathcal{A}} . s_\beta\right) \quad \dots (3')$$

$$(\mathcal{A} \otimes \beta)^c = \left(s_{\mathcal{A}} . s_\beta, i_{\mathcal{A}} . i_\beta, \sqrt[n]{d_{\mathcal{A}}^n + d_\beta^n - d_{\mathcal{A}}^n . d_\beta^n}\right)^c$$

$$= \left(\sqrt[n]{d_{\mathcal{A}}^n + d_\beta^n - d_{\mathcal{A}}^n . d_\beta^n}, i_{\mathcal{A}} . i_\beta, s_{\mathcal{A}} . s_\beta\right) \quad \dots (4')$$

From $(3')$ and $(4')$, we have $\mathcal{A}^c \oplus \beta^c = (\mathcal{A} \otimes \beta)^c$

The newly defined operations successfully satisfied the basic set theoretic properties showing their fitness and viability.

4. T-Spherical Averaging Aggregation Operators

In this section, the averaging operators for TSFSs are developed including TSFWA operators, TSFOWA operators and TSFHA operators. We investigated the basic properties of these operators like monotonicity, idempotency and boundedness etc. The fitness of these AOs is validated using mathematical induction. These operations are applied in in MADM problem in section 5. Further, in our study of aggregation theory $w = \left(w_1, w_2, w_{3,} \dots w_m\right)^{\mathcal{T}}$ will denote the weight vector (WV) of s, i, d and r for $j = 1, 2, 3, \dots m$ such that $w_j > 0$ and $\sum_{j=1}^{m} w_j = 1$.

4.1. Averaging Operators

In this subsection, the weighted averaging operators for TSFSs are defined and their fitness is checked using induction. The properties of averaging operators are also investigated. The operations developed here generalizes the operations of IFS [31], PyFSs [9] and PFSs [13] (For proof please see Remark 8 in section 7)

Definition 9: For some TSFNs $\mathcal{T}_j$, the TSFWA operator is a mapping defined as:

$$\mathcal{T}SFW\mathcal{A}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \sum_{j=1}^{m} w_j \mathcal{T}_j$$

Theorem 2: The aggregated value of some TSFNs $\mathcal{T}_j$ using TSFWA operator is a TSFN and is given by:

$$\mathcal{T}SFW\mathcal{A}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \left(\sqrt[n]{1 - \prod_{j=1}^{m}\left(1 - s_j^n\right)^{w_j}}, \prod_{j=1}^{m}\left(i_j\right)^{w_j}, \prod_{j=1}^{m}\left(d_j\right)^{w_j}\right)$$

Proof: The result is proved using mathematical induction:

For $m = 2$

$w_1\mathcal{T}_1 = \left(\sqrt[n]{1 - (1 - s_1^n)^{w_1}}, (i_1)^{w_1}, (d_1)^{w_1}\right)$ and $w_2\mathcal{T}_2 = \left(\sqrt[n]{1 - (1 - s_2^n)^{w_2}}, (i_2)^{w_2}, (d_2)^{w_2}\right)$

$$w_1\mathcal{T}_1 \oplus w_2\mathcal{T}_2 = \left(\sqrt[n]{1 - (1 - s_1^n)^{w_1}}, (i_1)^{w_1}, (d_1)^{w_1}\right) \oplus \left(\sqrt[n]{1 - (1 - s_2^n)^{w_2}}, (i_2)^{w_2}, (d_2)^{w_2}\right)$$

$$= \left(\sqrt[n]{\left(\sqrt[n]{1-(1-s_1^n)^{w_1}}\right)^n + \left(\sqrt[n]{1-(1-s_2^n)^{w_2}}\right)^n - \sqrt[n]{1-(1-s_1^n)^{w_1}}.\sqrt[n]{1-(1-s_2^n)^{w_2}}}, (i_1)^{w_1}.(i_2)^{w_2}, (d_1)^{w_1}.(d_2)^{w_2} \right)$$

$$= \left(\sqrt[n]{1-(1-s_1^n)^{w_1}.(1-s_2^n)^{w_2}}, (i_1)^{w_1}.(i_2)^{w_2}, (d_1)^{w_1}.(d_2)^{w_2} \right)$$

$$= \left(\sqrt[n]{1-(1-s_1^n)^{w_1}.(1-s_2^n)^{w_2}}, (i_1)^{w_1}.(i_2)^{w_2}, (d_1)^{w_1}.(d_2)^{w_2} \right)$$

$$= \left(\sqrt[n]{1-\prod_{j=1}^{2}(1-s_j^n)^{w_j}}, \prod_{j=1}^{2}(i_j)^{w_j}, \prod_{j=1}^{2}(d_j)^{w_j} \right)$$

Assume that result is true for $m = k$ i.e.

$$\mathcal{TSFWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_{\boldsymbol{k}}) = \left(\sqrt[n]{1-\prod_{j=1}^{\boldsymbol{k}}(1-s_j^n)^{w_j}}, \prod_{j=1}^{\boldsymbol{k}}(i_j)^{w_j}, \prod_{j=1}^{\boldsymbol{k}}(d_j)^{w_j} \right)$$

To prove this result for $m = k+1$. Consider

$$\mathcal{TSFWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_{\boldsymbol{k}}, \mathcal{T}_{k+1}) = \sum_{j=1}^{k+1} w_j \mathcal{T}_j = \sum_{j=1}^{k} w_j \mathcal{T}_j \oplus w_{k+1}\mathcal{T}_{k+1}$$

$$= \left(\sqrt[n]{1-\prod_{j=1}^{\boldsymbol{k}}(1-s_j^n)^{w_j}}, \prod_{j=1}^{\boldsymbol{k}}(i_j)^{w_j}, \prod_{j=1}^{\boldsymbol{k}}(d_j)^{w_j} \right) \oplus \left(\left(\sqrt[n]{1-(1-s_2^n)^{w_{k+1}}}\right), (i_2)^{w_{k+1}}, (d_2)^{w_{k+1}} \right)$$

Proceeding like we did in **Step 1**.

$$= \left(\sqrt[n]{1-\prod_{j=1}^{\boldsymbol{k}+1}(1-s_j^n)^{w_j}}, \prod_{j=1}^{\boldsymbol{k}+1}(i_j)^{w_j}, \prod_{j=1}^{\boldsymbol{k}+1}(d_j)^{w_j} \right)$$

Hence the result holds for $m = k+1$.

In [9, 13, 31] some important features of AOs are discussed showing their fitness and strength. Here we listed the characteristics of TSFWA operators.

Theorem 3: (Characteristics of TSFWA Operators)

1. (Idempotency) If for all $j = 1, 2, 3, \ldots m$, $\mathcal{T}_j = \mathcal{T} = (s, i, d)$. Then

$$\mathcal{TSFWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_m) = \mathcal{T}$$

2. (Boundedness) If $\mathcal{T}^- = \left(\min_j s_j, \max_j i_j, \max_j d_j\right)$ and $\mathcal{T}^+ = \left(\max_j s_j, \min_j i_j, \min_j d_j\right)$. Then

$$\mathcal{T}^{-} \leq \mathcal{TSFWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_m) \leq \mathcal{T}^{+}$$

3. (Monotonicity) Let $\mathcal{T}_j = \left(s_{\mathcal{T}_j}, i_{\mathcal{T}_j}, d_{\mathcal{T}_j}\right)$ and $P_j = \left(s_{P_j}, i_{P_j}, d_{P_j}\right)$ be two TSFNs such that $\mathcal{T}_j \leq P_j$ $\forall\, j$. Then

$$\mathcal{TSFWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_m) \leq \mathcal{TSFWA}(P_1, P_2, P_3 \ldots P_m)$$

4. (Shift Invariance) For another TSFN $P = (s_P, i_P, d_P)$

$$\mathcal{TSFWA}(\mathcal{T}_1 + P, \mathcal{T}_2 + P, \mathcal{T}_3 + P, \ldots \mathcal{T}_m + P) \leq \mathcal{TSFWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_m) \oplus P$$

5. (Homogeneity) For $\lambda > 0$

$$\mathcal{TSFWA}(\lambda\mathcal{T}_1, \lambda\mathcal{T}_2, \lambda\mathcal{T}_3 \ldots \lambda\mathcal{T}_m) = \lambda.\mathcal{TSFWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_m)$$

Proof: The proofs of these results are obvious so are omitted here.

The following example illustrates how TSFWA operators works.

Example 1: Consider for $n = 2$, we have three TSFNs $\mathcal{T}_1 = (0.67, 0.34, 0.58)$, $\mathcal{T}_2 = (0.43, 0.59, 0.31)$ and $\mathcal{T}_3 = (0.78, 0.63, 0.48)$ and $w = (0.5, 0.3, 0.2)^{\mathcal{T}}$ be the WV. Then

$$\mathcal{TSFWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$$

$$= \Big(\sqrt[2]{1 - ((1 - 0.67^2)^{0.5})((1 - 0.43^2)^{0.3})((1 - 0.78^2)^{0.2})}, (0.34^2)^{0.5}(0.59^2)^{0.3}(0.63^2)^{0.2}, (0.58^2)^{0.5}(0.31^2)^{0.3}(0.2^2)^{0.2}\Big)$$

$$\mathcal{TSFWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) = (0.64898, 0.453799, 0.374583)$$

4.2. Ordered Weighted and Hybrid Averaging Operators

In previous section, TSFWA operators are developed and exemplified. In TSFWA operators, the TSFNs are weighted. Sometimes in MADM problems, when we need to weight the ordered position of the TSFN, then we need to develop TSFOWA operator and when we need to weight the TSFNs as well as its ordered position we use the concept of TSFHA operator. Therefore, in this section we developed TSFOWA operators and TSFHA operators and explained them with the help of examples.

Definition 10: For some TSFNs $\mathcal{T}_j$, the TSFOWA operator is a mapping defined as:

$$\mathcal{TSFOWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_m) = \sum_{j=1}^{m} w_j \mathcal{T}_{\sigma(j)}$$

Where $\mathcal{T}_{\sigma(j)}$ is the j[th] largest of TSFNs $\mathcal{T}_j$.

Theorem 4: The aggregated value of some TSFNs $\mathcal{T}_j$ using TSFOWA operator is a TSFN and is given by:

$$\mathcal{TSFOWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_m) = \left(\sqrt[n]{1 - \prod_{j=1}^{m}\left(1 - s_{\sigma(j)}^{n}\right)^{w_j}}, \prod_{j=1}^{m}\left(i_{\sigma(j)}\right)^{w_j}, \prod_{j=1}^{m}\left(d_{\sigma(j)}\right)^{w_j}\right)$$

Proof: The proof is similar to that of Theorem 2.

Remark 7: Some Special Cases:

1. For $w = (1, 0, 0, \dots 0)^{\mathcal{T}}$, $\mathcal{TSFOWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \max\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m\}$
2. For $w = (0, 0, 0, \dots 1)^{\mathcal{T}}$, $\mathcal{TSFOWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \min\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m\}$
3. For $w_j = 1$ or 0, $\mathcal{TSFOWA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \mathcal{T}_{\sigma(j)}$ where $\mathcal{T}_{\sigma(j)}$ is the jth largest of TSFNs $\mathcal{T}_j$.

Example 2: We solve Example 1 using TSFOWA operators. To do that, first we need to calculate the score values of TSFNs to arrange them in ordered positions and then simply aggregate them using TSFOWA operator. The score values are: $\check{S}(\mathcal{T}_1) = 0.3045$, $\check{S}(\mathcal{T}_2) = 0.0888$, $\check{S}(\mathcal{T}_3) = 0.37$. Based on score values, the new ordered position of TSFNSs are:

$\mathcal{T}_{\sigma(1)} = \mathcal{T}_3 = (0.78, 0.63, 0.48)$

$\mathcal{T}_{\sigma(2)} = \mathcal{T}_1 = (0.67, 0.34, 0.58)$

$\mathcal{T}_{\sigma(3)} = \mathcal{T}_2 = (0.43, 0.59, 0.31)$

$\mathcal{TSFWA}(\mathcal{T}_3, \mathcal{T}_1, \mathcal{T}_2) = (0.705422, 0.516755, 0.410042)$

Definition 11: For some TSFNs $\mathcal{T}_j$, the TSFHA operator is a mapping defined as:

$$\mathcal{TSFHA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \sum_{j=1}^{m} w_j \dot{\mathcal{T}}_{\sigma(j)}$$

Where $\dot{\mathcal{T}}_{\sigma(j)}$ is the jth largest of TSFNs $\dot{\mathcal{T}}_j$ and $\dot{\mathcal{T}}_j = m\omega_j \mathcal{T}_j$ such that m is the number of TSFNs and $\omega = (\omega_1, \omega_2, \omega_3 \dots \omega_n)^{\mathcal{T}}$ is the WV of $\mathcal{T}_j$.

While applying TSFHA operator, we first determine $\dot{\mathcal{T}}_j = m\omega_j \mathcal{T}_j$ using the WV $\omega = (\omega_1, \omega_2, \omega_3 \dots \omega_n)^{\mathcal{T}}$. Then the weighted TSFNs $\dot{\mathcal{T}}_j$ are rearranged where $\dot{\mathcal{T}}_{\sigma(j)}$ is the jth largest of TSFNs $\dot{\mathcal{T}}_j$. Finally, the TSFHA operator is used to aggregate the TSFNs $\dot{\mathcal{T}}_j$. Using the basic operations of TSFNs, the following theorem is proposed.

Theorem 5: The aggregated value of some TSFNs $\mathcal{T}_j$ using TSFHA operator is a TSFN and is given by:

$$\mathcal{TSFHA}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \left(\sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - \dot{s}_{\sigma(j)}^{n}\right)^{w_j}}, \prod_{j=1}^{m} \left(\dot{i}_{\sigma(j)}\right)^{w_j}, \prod_{j=1}^{m} \left(\dot{d}_{\sigma(j)}\right)^{w_j} \right)$$

Proof: The proof is similar to that of Theorem 2.

Example 3: Consider for $n = 2$, we have three TSFNs $\mathcal{T}_1 = (0.67, 0.34, 0.58)$, $\mathcal{T}_2 = (0.43, 0.59, 0.31)$ and $\mathcal{T}_3 = (0.78, 0.63, 0.48)$ and $\omega = (0.4, 0.35, 0.25)^{\mathcal{T}}$ be the WV of given TSFNs while $w = (0.5, 0.3, 0.2)^{\mathcal{T}}$ is the aggregated associated weighted vector. Then

$$\dot{\mathcal{T}}_1 = 3\omega_1 \mathcal{T}_1 = 3 \times 0.4\, \mathcal{T}_1 = \left(\sqrt[2]{1 - (1 - 0.67^2)^{3\times 0.4}}, (0.34^2)^{3\times 0.4}, (0.58^2)^{3\times 0.4} \right) = (0.71, 0.27, 0.52)$$

Similarly, $\dot{\mathcal{T}}_2 = (0.44, 0.57, 0.29)$ and $\dot{\mathcal{T}}_3 = (0.71, 0.71, 0.58)$. Now we use score function to find the ordered position of $\dot{\mathcal{T}}_j$ as follows:

$\check{S}(\dot{\mathcal{T}}_1) = 0.2337$, $\check{S}(\dot{\mathcal{T}}_2) = 0.1095$, $\check{S}(\dot{\mathcal{T}}_3) = 0.1677$. Based on score values, the new ordered position of TSFNSs are:

$\dot{\mathcal{T}}_{\sigma(1)} = \dot{\mathcal{T}}_1 = (0.7, 0.08, 0.27)$

$\dot{\mathcal{T}}_{\sigma(2)} = \dot{\mathcal{T}}_3 = (0.7, 0.5, 0.33)$

$\dot{\mathcal{T}}_{\sigma(3)} = \dot{\mathcal{T}}_2 = (0.44, 0.33, 0.08)$

$\mathcal{T}SFW\mathcal{A}(\dot{\mathcal{T}}_1, \dot{\mathcal{T}}_3, \dot{\mathcal{T}}_2) = (0.65286, 0.409917, 0.446074)$

Theorem 6: If we assume the WV w to be $w = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^{\mathcal{T}}$. Then the TSFHA operator reduces to TSFWA operator.

Proof: As $\dot{\mathcal{T}}_j = m\omega_j\mathcal{T}_j$ and $w = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^{\mathcal{T}}$. So $w_j\dot{\mathcal{T}}_j = \omega_j\mathcal{T}_j$ and

$$\mathcal{T}SFH\mathcal{A}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \sum_{j=1}^{m} w_j\dot{\mathcal{T}}_{\sigma(j)} = \sum_{j=1}^{m} \omega_j\mathcal{T}_j = \mathcal{T}SFW\mathcal{A}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m)$$

Theorem 7: If we assume the WV ω to be $\omega = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^{\mathcal{T}}$. Then the TSFHA operator reduces to TSFOWA operator.

Proof: Straightforward.

5. T-Spherical Fuzzy Geometric Operators

In this section, the geometric operators for TSFSs are developed including TSFWG operator, TSFOWG operator and TSFHG operator. We investigated the basic properties of these operators like monotonicity, idempotency and boundedness etc. The fitness of these AOs is checked using mathematical induction.

5.1. Weighted Geometric Operators

In this subsection, the weighted geometric operator for TSFSs is defined and its fitness is checked using mathematical induction. The properties of averaging operator are also investigated. The operations developed here generalizes the operations of IFS, PyFSs and PFSs (For proof please see Remark 8 in section 7)

Definition 12: For some TSFNs $\mathcal{T}_j$, the TSFWG operator is a mapping defined as:

$$\mathcal{T}SFWG(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \prod_{j=1}^{m} \mathcal{T}_j^{w_j}$$

Theorem 8: The aggregated value of some TSFNs $\mathcal{T}_j$ using TSFWG operator is a TSFN and is given by:

$$\mathcal{T}SFWG(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \left(\prod_{j=1}^{m}(s_j)^{w_j}, \prod_{j=1}^{m}(i_j)^{w_j}, \sqrt[n]{1 - \prod_{j=1}^{m}(1 - d_j^n)^{w_j}}\right)$$

Proof: The result is proved using mathematical induction:

For $m = 2$

$w_1\mathcal{T}_1 = \left((s_1)^{w_1}, (i_1)^{w_1}, \sqrt[n]{1-(1-d_1^n)^{w_1}}\right)$ and $w_2\mathcal{T}_2 = \left((s_2)^{w_2}, (i_2)^{w_2}, \sqrt[n]{1-(1-d_2^n)^{w_2}}\right)$

$$w_1\mathcal{T}_1 \otimes w_2\mathcal{T}_2 = \left((s_1)^{w_1}, (i_1)^{w_1}, \sqrt[n]{1-(1-d_1^n)^{w_1}}\right) \otimes \left((s_2)^{w_2}, (i_2)^{w_2}, \sqrt[n]{1-(1-d_2^n)^{w_2}}\right)$$

$$= \left(\begin{array}{c} (s_1)^{w_1}.(s_2)^{w_2}, (i_1)^{w_1}.(i_2)^{w_2}, \\ \sqrt[n]{\left(\sqrt[n]{1-(1-d_1^n)^{w_1}}\right)^n + \left(\sqrt[n]{1-(1-d_2^n)^{w_2}}\right)^n - \sqrt[n]{1-(1-d_1^n)^{w_1}}.\sqrt[n]{1-(1-d_2^n)^{w_2}}} \end{array}\right)$$

$$= \left((s_1)^{w_1}.(s_2)^{w_2}, (i_1)^{w_1}.(i_2)^{w_2}, \sqrt[n]{1-(1-d_1^n)^{w_1}.(1-d_2^n)^{w_2}}\right)$$

$$= \left(\prod_{j=1}^{2}(s_j)^{w_j}, \prod_{j=1}^{2}(i_j)^{w_j}, \sqrt[n]{1-\prod_{j=1}^{2}\left(1-d_j^n\right)^{w_j}}\right)$$

True for $m = 2$. Assume that result holds for $m = k$ i.e.

$$\mathcal{T}SFWG(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_k) = \left(\prod_{j=1}^{k}(s_j)^{w_j}, \prod_{j=1}^{k}(i_j)^{w_j}, \sqrt[n]{1-\prod_{j=1}^{k}\left(1-d_j^n\right)^{w_j}}\right)$$

To prove for $m = k+1$. Consider

$$\mathcal{T}SFWG(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_{\boldsymbol{k}}, \mathcal{T}_{k+1}) = \prod_{j=1}^{k+1}\mathcal{T}_j^{w_j} = \prod_{j=1}^{k}\mathcal{T}_j^{w_j} \otimes \mathcal{T}_{k+1}^{w_{k+1}}$$

$$= \left(\prod_{j=1}^{k}(s_j)^{w_j}, \prod_{j=1}^{k}(i_j)^{w_j}, \sqrt[n]{1-\prod_{j=1}^{k}\left(1-d_j^n\right)^{w_j}}\right) \oplus \left((s_2)^{w_{k+1}}, (i_2)^{w_{k+1}}, \left(\sqrt[n]{1-(1-d_2^n)^{w_{k+1}}}\right)\right)$$

Finally,

$$= \left(\prod_{j=1}^{k+1}(s_j)^{w_j}, \prod_{j=1}^{k+1}(i_j)^{w_j}, \sqrt[n]{1-\prod_{j=1}^{k+1}\left(1-d_j^n\right)^{w_j}}\right)$$

Hence the result holds for $m = k+1$.

In [9, 13, 31] some important features of AOs are discussed showing their validity. Here we listed the characteristics of TSFWG operators.

Theorem 9: (Characteristics of TSFWG Operators)

1. (Idempotency) If for all $j = 1, 2, 3, \ldots m$, $\mathcal{T}_j = \mathcal{T} = (s, i, d)$. Then

$$\mathcal{T}SFWG(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_m) = \mathcal{T}$$

2. (Boundedness) If $\mathcal{T}^- = \left(\min_j s_j, \max_j i_j, \max_j d_j\right)$ and $\mathcal{T}^+ = \left(\max_j s_j, \min_j i_j, \min_j d_j\right)$. Then

$$\mathcal{T}^- \leq \mathcal{T}SFWG(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_m) \leq \mathcal{T}^+$$

3. (Monotonicity) Let $\mathcal{T}_j = \left(s_{\mathcal{T}_j}, i_{\mathcal{T}_j}, d_{\mathcal{T}_j}\right)$ and $P_j = \left(s_{P_j}, i_{P_j}, d_{P_j}\right)$ be two TSFNs such that $\mathcal{T}_j \leq P_j$ $\forall\, j$. Then

$$\mathcal{T}SFWG(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_m) \leq \mathcal{T}SFWG(P_1, P_2, P_3 \ldots P_m)$$

Proof: The proofs of these results are obvious so are omitted here.

Example 4: Consider for $n = 2$, we have three TSFNs $\mathcal{T}_1 = (0.67, 0.34, 0.58)$, $\mathcal{T}_2 = (0.43, 0.59, 0.31)$ and $\mathcal{T}_3 = (0.78, 0.63, 0.48)$ and $w = (0.5, 0.3, 0.2)^{\mathcal{T}}$ be the WV. Then

$\mathcal{T}SFWG(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$

$$= \begin{pmatrix} (0.58^2)^{0.5}(0.43^2)^{0.3}(0.78^2)^{0.2}, (0.34^2)^{0.5}(0.59^2)^{0.3}(0.63^2)^{0.2}, \\ \sqrt[2]{1-((1-0.58^2)^{0.5})((1-0.31^2)^{0.3})((1-0.48^2)^{0.2})} \end{pmatrix}$$

$$\mathcal{T}SFWG(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) = (0.60464, 0.453799, 0.385256)$$

5.2. Ordered Weighted Geometric and Hybrid Geometric Operators

In TSFWG operators, the TSFNs are weighted. Sometimes in MADM problems, when we need to weight the ordered position of the TSFNs, then we need to develop the ordered weighted geometric operators and when we need to weight the TSFNs as well as its ordered position, we used the concept of hybrid geometric operators. Therefore, in this section we developed TSFOWG operators and TSFHG operators.

Definition 13: For some TSFNs $\mathcal{T}_j$, the TSFOWG operator is a mapping defined as:

$$\mathcal{T}SFOWG(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_m) = \prod_{j=1}^{m} \mathcal{T}_{\sigma(j)}^{w_j}$$

Where $\mathcal{T}_{\sigma(j)}$ is the jth largest of TSFNs $\mathcal{T}_j$.

Theorem 10: The aggregated value of some TSFNs $\mathcal{T}_j$ using TSFOWG operator is a TSFN and is given by:

$$\mathcal{T}SFOWG(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \ldots \mathcal{T}_m) = \left(\prod_{j=1}^{m}\left(s_{\sigma(j)}\right)^{w_j}, \prod_{j=1}^{m}\left(i_{\sigma(j)}\right)^{w_j}, \sqrt[n]{1-\prod_{j=1}^{m}\left(1-d_{\sigma(j)}^{n}\right)^{w_j}}\right)$$

Proof: The proof is similar to that of Theorem 2.

Example 5: We solve Example 1 using TSFOWG operators. To do that, first we need to calculate the score values of TSFNs to arrange them in ordered positions and then simply aggregate them using TSFOWG operator. The score values are: $\check{S}(\mathcal{T}_1) = 0.3045$, $\check{S}(\mathcal{T}_2) = 0.0888$, $\check{S}(\mathcal{T}_3) = 0.37$. Based on score values, the new ordered position of TSFNSs are:

$\mathcal{T}_{\sigma(1)} = \mathcal{T}_3 = (0.78, 0.63, 0.48)$

$\mathcal{T}_{\sigma(2)} = \mathcal{T}_1 = (0.67, 0.34, 0.58)$

$\mathcal{T}_{\sigma(3)} = \mathcal{T}_2 = (0.43, 0.59, 0.31)$

$$\mathcal{TSFWG}(\mathcal{T}_3, \mathcal{T}_1, \mathcal{T}_2) = (0.856729, 0.516755, 0.322408)$$

Definition 14: For some TSFNs $\mathcal{T}_j$, the TSFHG operator is a mapping defined as:

$$\mathcal{TSFHG}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \prod_{j=1}^{m} \dot{\mathcal{T}}_{\sigma(j)}^{w_j}$$

Where $\dot{\mathcal{T}}_{\sigma(j)}$ is the j[th] largest of TSFNs $\dot{\mathcal{T}}_j$ and $\dot{\mathcal{T}}_j = \mathcal{T}_j^{m\omega_j}$ such that m is the number of TSFNs and $\omega = (\omega_1, \omega_2, \omega_3 \dots \omega_n)^{\mathcal{T}}$ is the WV of $\mathcal{T}_j$.

While applying TSFHG operator, we first determine $\dot{\mathcal{T}}_j = \mathcal{T}_j^{m\omega_j}$ using the WV $\omega = (\omega_1, \omega_2, \omega_3 \dots \omega_n)^{\mathcal{T}}$. Then the weighted TSFNs $\dot{\mathcal{T}}_j$ are rearranged where $\dot{\mathcal{T}}_{\sigma(j)}$ is the j[th] largest of TSFNs $\dot{\mathcal{T}}_j$. Finally, the TSFHG operator is used to aggregate the TSFNs $\dot{\mathcal{T}}_j$. Using the basic operations of TSFNs, the following theorem is proposed.

Theorem 11: The aggregated value of some TSFNs $\mathcal{T}_j$ using TSFHG operator is a TSFN and is given by:

$$\mathcal{TSFHG}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \left(\prod_{j=1}^{m} \left(\dot{s}_{\sigma(j)}\right)^{w_j}, \prod_{j=1}^{m} \left(i_{\sigma(j)}\right)^{w_j}, \sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - \dot{d}_{\sigma(j)}^{n}\right)^{w_j}} \right)$$

Proof: The proof is similar to that of Theorem 2.

The following example illustrate how hybrid operators works.

Example 6: Consider for $n = 2$, we have three TSFNs $\mathcal{T}_1 = (0.67, 0.34, 0.58)$, $\mathcal{T}_2 = (0.43, 0.59, 0.31)$ and $\mathcal{T}_3 = (0.78, 0.63, 0.48)$ and $\omega = (0.4, 0.35, 0.25)^{\mathcal{T}}$ be the WV of given TSFNs while $w = (0.5, 0.3, 0.2)^{\mathcal{T}}$ is the aggregated associated weighted vector. Then

$\dot{\mathcal{T}}_1 = \mathcal{T}_1^{3\omega_1} = \mathcal{T}_1^{3\times 0.4}$

$= \left((0.67^2)^{3\times 0.4}, (0.34^2)^{3\times 0.4}, \sqrt[2]{1 - (1 - 0.58^2)^{3\times 0.4}} \right)$

$= (0.618429, 0.274015, 0.623421)$

Similarly, $\dot{\mathcal{T}}_2 = (0.412232, 0.574638, 0.317261)$ and $\dot{\mathcal{T}}_3 = (0.829986, 0.70714, 0.422288)$. Now we use score function to find the ordered position of $\dot{\mathcal{T}}_j$ as follows:

$\check{S}(\dot{\mathcal{T}}_1) = 0.006199$, $\check{S}(\dot{\mathcal{T}}_2) = -0.06928$, $\check{S}(\dot{\mathcal{T}}_3) = -0.51055$. Based on score values, the new ordered position of TSFNSs are:

$\dot{\mathcal{T}}_{\sigma(1)} = \dot{\mathcal{T}}_1 = (0.618429, 0.274015, 0.623421)$

$\dot{\mathcal{T}}_{\sigma(2)} = \dot{\mathcal{T}}_3 = (0.829986, 0.70714, 0.422288)$

$\dot{\mathcal{T}}_{\sigma(3)} = \dot{\mathcal{T}}_2 = (0.412232, 0.574638, 0.317261)$

$\mathcal{TSFWG}(\dot{\mathcal{T}}_1, \dot{\mathcal{T}}_3, \dot{\mathcal{T}}_2) = (0.483282, 0.41902, 0.683548)$

Theorem 12: If we assume the WV w to be $w = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^{\mathcal{T}}$. Then the TSFHG operator reduces to TSFWG operator.

Proof: As $\dot{\mathcal{T}}_j = \mathcal{T}_j^{m\omega_j}$ and $w = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^{\mathcal{T}}$. So $\dot{\mathcal{T}}_j^{w_j} = \mathcal{T}_j^{\omega_j}$ and

$$\mathcal{TSFHG}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \prod_{j=1}^{m} \dot{\mathcal{T}}_{\sigma(j)}^{w_j} = \prod_{j=1}^{m} \mathcal{T}_j^{w_j} = \mathcal{TSFWG}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m)$$

Theorem 13: If we assume the WV ω to be $\omega = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^{\mathcal{T}}$. Then the TSFHG operator reduces to TSFOWG operator.

Proof: Straightforward.

6. Multi-Attribute Decision Making

In MADM process, we rank a set of alternatives $\mathcal{A}_j$ based on some attributes K_j having weight vector w. In this phenomenon, a panel of decision makers evaluated the alternatives $\mathcal{A}_j$ and provided their information in the form of TSFNs i.e. in a decision matrix containing TSFNs. Then the different types of aggregators are utilized to aggregate the information for the evaluation of best alternatives. The steps of algorithm of MADM process are described below:

6.1. Algorithm:

Step 1: The information in the form of TSFNs under some attributes about the alternatives are gathered and a decision matrix is formed.

Step 2: In step 2, the data provided by the decision makers is aggregated using AOs of TSFNs.

Step 3: In step 3, the scores of the aggregated data using Definition 7 is obtained.

Step 4: The final step involves the ranking of score values and most suitable alternative is obtained.

To demonstrate the MADM algorithm, we present a numerical example adapted form [20]. The data provided in this example could not be handled using IFSs or PFSs which proves the novelty of proposed operators in the environment of TSFSs.

Example 7: A multinational company is designing its financial policy for the upcoming year about where to invest to get a potential profit. For this, the research department of the company came with four plans about where to invest after some initial screening. These four alternatives are, P_1: Asian Markets, P_2: Local Markets, P_3: European Markets and P_4: African Markets. The evaluation of suitable market to invest in is based on four attributes which are, $\mathcal{A}_1$: The growth perspective, $\mathcal{A}_2$: risk perspective, $\mathcal{A}_3$: political and social perspective and $\mathcal{A}_4$: environmental perspective. The WV is $w = (0.2, 0.1, 0.3, 0.4)^{\mathcal{T}}$. The stepwise demonstration of MADM process is as follows:

Step 1: The formation of decision matrix in Table 1. Note that all the data provided in Table 1 are purely TSFNs for $n = 3$.

	A_1	A_2	A_3	A_4

$\boldsymbol{P_1}$	$(0.53, 0.33, 0.38)$	$(0.65,0.24,0.74)$	$(0.61, 0.39, 0.45)$	$(0.55, 0.88, 0.29)$
$\boldsymbol{P_2}$	$(0.40, 0.71, 0.15)$	$(0.48, 0.46,0.67)$	$(0.69, 0.46, 0.29)$	$(0.61, 0.73, 0.43)$
$\boldsymbol{P_3}$	$(0.33, 0.53, 0.79)$	$(0.71, 0.49, 0.16)$	$(0.53, 0.39, 0.84)$	$(0.50,0.90, 0.01)$
$\boldsymbol{P_4}$	$(0.64, 0.38,0.73)$	$(0.33,0.64,0.76)$	$(0.27,0.89,0.07)$	$(0.74, 0.36,0.19)$

Table 1 (Decision Matrix)

Step 2: The data provided by the decision makers in Table 1 is aggregated using TSFWA operators in this step. The steps involved are already explained, here we just provided the results which are:

$$\mathcal{T}_1 = \mathcal{T}SFW\mathcal{A}(\mathcal{T}_{11}, \mathcal{T}_{12}, \mathcal{T}_{13}, \mathcal{T}_{14}) = (0.548023, 0.49755, 0.383533)$$

$$\mathcal{T}_2 = \mathcal{T}SFW\mathcal{A}(\mathcal{T}_{21}, \mathcal{T}_{22}, \mathcal{T}_{23}, \mathcal{T}_{24}) = (0.602926, 0.603509, 0.323543)$$

$$\mathcal{T}_3 = \mathcal{T}SFW\mathcal{A}(\mathcal{T}_{31}, \mathcal{T}_{32}, \mathcal{T}_{33}, \mathcal{T}_{34}) = (0.521966, 0.592777, 0.11946)$$

$$\mathcal{T}_4 = \mathcal{T}SFW\mathcal{A}(\mathcal{T}_{41}, \mathcal{T}_{42}, \mathcal{T}_{43}, \mathcal{T}_{44}) = (0.623935, 0.505723, 0.211727)$$

Step 3: Now we compute the score values of the date obtained in Step 2.

$$\check{S}(\mathcal{T}_1) = (0.548023)^3 - (0.056417)^3 = 0.10817$$

$$\check{S}(\mathcal{T}_2) = (0.602926)^3 - (0.033869)^3 = 0.185307$$

$$\check{S}(\mathcal{T}_3) = (0.521966)^3 - (0.001705)^3 = 0.140504$$

$$\check{S}(\mathcal{T}_4) = (0.623935)^3 - (0.009491)^3 = 0.233403$$

Step 4: Step 4 involves the comparison of score values obtained in Step 3. The comparison is as follows:

$$\check{S}(\mathcal{T}_4) > \check{S}(\mathcal{T}_2) > \check{S}(\mathcal{T}_3) > \check{S}(\mathcal{T}_1)$$

Clearly, the score of $\mathcal{T}_4$ is greater among all so the firm needed to go with policy 4 i.e. to invest in African markets according to the evaluation of the data using TSFWA operators. Such type of decision making could be very helpful in management sciences problems, economic problems and problems of engineering and computer sciences where one need to choose among some alternatives based on expert's opinion.

7. Comparative Study and Advantages:

In this section we are about to establish a comparative study of proposed study and existing work which will demonstrate the diverse nature of TSFSs and its AOs along with the limitations of IFSs, Pythagorean FSs and PFSs. FSs, IFSs, Pythagorean FSs and PFSs are all special cases of TSFS as described in [18]. Here we are interested in showing that the proposed AOs are the generalizations of AOs of IFSs [31] Pythagorean fuzzy sets [9] and PFSs [13]. The following remark will explain how the AOs of IFSs, Pythagorean FSs and PFSs becomes particular cases of T-spherical fuzzy AOs.

Remark 8: Consider the TSFWA and TSFWG operator as follows:

$$\mathcal{T}SFW\mathcal{A}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \left(\sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - s_j^n\right)^{w_j}}, \prod_{j=1}^{m} \left(i_j\right)^{w_j}, \prod_{j=1}^{m} \left(d_j\right)^{w_j} \right)$$

$$\mathcal{T}SFWG(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3 \dots \mathcal{T}_m) = \left(\prod_{j=1}^{m}(s_j)^{w_j}, \prod_{j=1}^{m}(i_j)^{w_j}, \sqrt[n]{1-\prod_{j=1}^{m}(1-d_j^n)^{w_j}}\right)$$

If we take the $n = 2$, we obtained weighted averaging and geometric operators of SFSs given as:

$$SFW\mathcal{A}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3 \dots \mathcal{T}_m) = \left(\sqrt[2]{1-\prod_{j=1}^{m}(1-s_j^2)^{w_j}}, \prod_{j=1}^{m}(i_j)^{w_j}, \prod_{j=1}^{m}(d_j)^{w_j}\right)$$

$$SFWG(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3 \dots \mathcal{T}_m) = \left(\prod_{j=1}^{m}(s_j)^{w_j}, \prod_{j=1}^{m}(i_j)^{w_j}, \sqrt[2]{1-\prod_{j=1}^{m}(1-d_j^2)^{w_j}}\right)$$

If we take the $n = 1$, we obtained weighted averaging and geometric operator of PFSs proposed by Garg [13] and are given as follows:

$$PFW\mathcal{A}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3 \dots \mathcal{T}_m) = \left(1-\prod_{j=1}^{m}(1-s_j)^{w_j}, \prod_{j=1}^{m}(i_j)^{w_j}, \prod_{j=1}^{m}(d_j)^{w_j}\right)$$

$$PFWG(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3 \dots \mathcal{T}_m) = \left(\prod_{j=1}^{m}(s_j)^{w_j}, \prod_{j=1}^{m}(i_j)^{w_j}, 1-\prod_{j=1}^{m}(1-d_j)^{w_j}\right)$$

If we take the $n = 2$ and $i = 0$, we obtained averaging and geometric operators of PyFSs developed by [9] and are given as follows:

$$PyFW\mathcal{A}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3 \dots \mathcal{T}_m) = \left(\sqrt[2]{1-\prod_{j=1}^{m}(1-s_j^2)^{w_j}}, \prod_{j=1}^{m}(d_j)^{w_j}\right)$$

$$PyFWG(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3 \dots \mathcal{T}_m) = \left(\prod_{j=1}^{m}(s_j)^{w_j}, \sqrt[2]{1-\prod_{j=1}^{m}(1-d_j^2)^{w_j}}\right)$$

If we take the $n = 1$ and $i = 0$, we obtained averaging and geometric operators defined in [31] and are given as follows:

$$IFW\mathcal{A}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3 \dots \mathcal{T}_m) = \left(1-\prod_{j=1}^{m}(1-s_j)^{w_j}, \prod_{j=1}^{m}(d_j)^{w_j}\right)$$

$$IFWG(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3 \dots \mathcal{T}_m) = \left(\prod_{j=1}^{m}(s_j)^{w_j}, 1-\prod_{j=1}^{m}(1-d_j)^{w_j}\right)$$

Therefore, it is proved that the AOs proposed in this manuscript are the generalizations of AOs of IFSs, Pythagorean FSs and PFSs and can handle the data which the existing tools could not.

As described in first section of the manuscript, that the theory of IFSs and Pythagorean FSs can only deal with situations where we face two types of options. These types of structures failed to describe the voting phenomena or situations where one has more than two opinions as described in [10, 11]. Also, Mahmood at all [18] pointed out towards the limitation of IFSs, Pythagorean FSs and PFSs and proposed SFSs and TSFSs that can handle not only the data provided in existing environments but also the data where the existing structures failed to be applied as discussed in [18, 19]. In view of all these facts, it is claimed that the aggregation theory proposed in this manuscript is better than the theory that is already exist and can model human opinion more effectively.

Further, we show that the proposed AOs of TSFSs can be applied to solve the problems lying in the environment of IFSs, PyFSs and PFSs etc. For this purpose, we consider Example 7 and dropped the abstinence degree from each triplet of Table 1. The new decision matrix obtained is provided in Table 2.

	A_1	A_2	A_3	A_4
P_1	$(0.53, 0.38)$	$(0.65, 0.74)$	$(0.61, 0.45)$	$(0.55, 0.29)$
P_2	$(0.40, 0.15)$	$(0.48, 0.67)$	$(0.69, 0.29)$	$(0.61, 0.43)$
P_3	$(0.33, 0.79)$	$(0.71, 0.16)$	$(0.53, 0.84)$	$(0.50, 0.01)$
P_4	$(0.64, 0.73)$	$(0.33, 0.76)$	$(0.27,0.07)$	$(0.74, 0.19)$

Table 2 (Decision Matrix after dropping the abstinence grade)

The information obtained after dropping the abstinence grade are now PyFNs as for all the duplets, the sum of square of the membership and non-membership grade is less than 1. Therefore, we use the following special case of the Remark 8 to aggregate the information obtained in Table 2.

$$PFW\mathcal{A}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \dots \mathcal{T}_m) = \left(1 - \prod_{j=1}^{m}(1 - s_j)^{w_j}, \prod_{j=1}^{m}(i_j)^{w_j}, \prod_{j=1}^{m}(d_j)^{w_j}\right)$$

The aggregated results are

$$\mathcal{T}_1 = (0.548023, 0.383533)$$

$$\mathcal{T}_2 = (0.602926, 0.323543)$$

$$\mathcal{T}_3 = (0.521966, 0.11946)$$

$$\mathcal{T}_4 = (0.623935, 0.211727)$$

To find the most suitable alternative, we use the score function and the score values are given by:

$$\check{S}(\mathcal{T}_1) = (0.548023)^3 - (0.383533)^3 = 0.10817$$

$$\check{S}(\mathcal{T}_2) = (0.602926)^3 - (0.323543)^3 = 0.185307$$

$$\check{S}(\mathcal{T}_3) = (0.521966)^3 - (0.11946)^3 = 0.140504$$

$$\check{S}(\mathcal{T}_4) = (0.623935)^3 - (0.11946)^3 = 0.233403$$

Based on the score values, we obtained the following ranking pattern

$$\check{S}(\mathcal{T}_4) > \check{S}(\mathcal{T}_2) > \check{S}(\mathcal{T}_3) > \check{S}(\mathcal{T}_1)$$

The result is consistent, and it shows that the proposed TSFWA operators are applicable in existing fuzzy environments and can solve any problem without any limitations.

8. Conclusion

In this manuscript, we developed some AOs for the novel concept of TSFS which a generalization of IFS is basically, Pythagorean FS and PFS. We discussed some basic theory related to IFS, PFS, SFS and TSFS including basic operations. Then we proposed some new operations for TSFSs, studied its several properties and proposed some results. Based on new operations, the concept of weighted averaging and geometric operators for TSFSs are proposed and their fitness is established using principle of mathematical induction. We also studied some characteristics like boundedness, idempotency, monotonicity and shift invariance. To meet some other situations of real-life problems, we defined ordered weighted averaging (geometric) and hybrid averaging (geometric) AOs for TSFSs as well. The defined AOs are illustrated with the help of some examples and a MADM problems is discussed considering the proposed work. Finally, a comparative study of current and new operations is established showing the significance of new work and some advantages of new proposed work. The current work can be extended to complex spherical and T-spherical fuzzy environments [53, 54] in near future. Further, the some other important applications such as transportation problems [55] and the evaluation of hospital performance [56] can be discussed in T-spherical fuzzy environment.

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