Annals of Optimization Theory and Practice

Volume 4, Number 1, 15-28
April 2021

DOI: 10.22121/AOTP.2021.259490.1058

Chain ratio-type estimator of finite population mean under two-phase sampling in systematic sampling scheme

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(Received: November 28, 2020 / Accepted: March 21, 2021)

Abstract In this article, a chain ratio-type estimator for estimating finite population means under two-phase sampling using systematic sampling has been proposed. The bias and mean square error up to a first-order approximation of the suggested estimator has been derived using the Taylor series expansion scheme. The conditions for which the proposed estimator outperformed other related estimators considered in the study have been established. An empirical study was also conducted to investigate the efficiency of the proposed estimator over some related existing estimators and the results revealed that the proposed estimator is more efficient.

Keywords Chain; Systematic sampling; Estimator; Mean squared error; Efficiency

1. Introduction

Systematic sampling is the most commonly used probability sampling due to its simplicity (Madow and Madow, 1944). It is also providing more efficient estimators than its counterparts such as simple random sampling and stratified random sampling for specific types of population (Cochran, 1946; Hajeck, 1959). The objective of survey sampler is to minimize errors either by formulating better estimators or proposing suitable sampling schemes (Singh and Solanki, 2012). It should be noted that there is likelihood of variations; that brings about the emphasizes on the need of better estimators which can be realized by proper use of auxiliary information when there is existence of strong positive or negative correlation between study and auxiliary variables. Often,

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population parameters of the auxiliary variable may be unknown while generating estimates of the population parameters of the variable under study, thus, result to double sampling in some cases which happens to be cost effective for obtaining a more reliable estimate than that of single-phase sampling.

The present work focuses on the application of power transformation in improving the efficiency of Abioye et al. (2019) on the modified ratio-cum-product estimators of population mean in linear systematic sampling under two-phase sampling scheme.

2.Nomenclatures used in the Study

Suppose the units of population of size are numbered from 1 to N in some order. To select a sample of size n units, if a unit is picked at random from the first k units and every k^{th} subsequent unit, then $= nk$. This selection procedure involved selecting a cluster out of k possible clusters and such that i^{th} cluster contained serially numbered units $i, i+k, i+2k, \dots, i+(n-1)k$. Let Y be the study variable and X, Z be auxiliary variables with values y_{ji}, x_{ji} and z_{ji} respectively $(j = 1,2,3,\dots,k), (i = 1,2,3,\dots,N_j)$

$\bar{Y} = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n} y_{ij}$ is the population mean for Y

$\bar{X} = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n} x_{ij}$ is the population mean for X

$\bar{Z} = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n} Z_{ij}$ is the population mean for Z

$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ is the sample mean for Y

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the sample mean for X

$\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$ is the sample mean for Z

$S_y^2 = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij}-\bar{Y})^2}{nk-1}$, $S_x^2 = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\bar{X})^2}{nk-1}$ and $S_z^2 = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}(z_{ij}-\bar{Z})^2}{nk-1}$ are variance for Y,X,Z respectively.

$S_{yx} = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ji}-\bar{Y})(x_{ji}-\bar{X})}{nk-1}$,$S_{yz} = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ji}-\bar{Y})(z_{ji}-\bar{Z})}{nk-1}andS_{xz} = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ji}-\bar{X})(z_{ji}-\bar{Z})}{nk-1}$ are covariance between Y,X,Z ,$c_y = \frac{s_y}{\bar{Y}}$, $c_x = \frac{s_x}{\bar{X}}$, $c_z = \frac{s_z}{\bar{Z}}$ are the coefficient of variation.

The conventional ratio and product estimators in systematic random sampling respectively as

$$t_1 = \bar{y}_{sys}\frac{\bar{X}}{\bar{x}_{sys}} \tag{1}$$

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$$t_2 = \bar{y}_{sys} \frac{\bar{x}_{sys}}{\bar{X}} \tag{2}$$

The biases and mean squared errors of the conventional ratio and product estimators up to first order approximation for t_1 and t_2 are given below;

$$Bias(t_1) = \lambda_2 \bar{Y}(B - D) \tag{3}$$

$$Bias(t_2) = \lambda_2 \bar{Y} D \tag{4}$$

$$MSE(t_1) = \lambda_2 \bar{Y}^2 (A + B - 2D) \tag{5}$$

$$MSE(t_2) = \lambda_2 \bar{Y}^2 (A + B + 2D) \tag{6}$$

where

$$A = C_y^2 \rho_y^*,\ B = C_x^2 \rho_x^*,\ D = \rho_{yx} C_y C_x \sqrt{\rho_y^* \rho_x^*}, \lambda_2 = \frac{(1-f)}{n}, f = \frac{n}{N}.$$

Both t_1 and t_2 give better estimate than t_0 if $\frac{B}{D} < 2$ and $\frac{B}{D} < -2$ respectively. Singh et al. (2011) suggested Modified ratio and product estimators in systematic random sampling. The suggested estimators are version of estimator of Bahl and Tuteja (1991). The suggested estimator and its mean squared error are given below;

$$t_3 = \bar{y}_{sys} \exp\left(\frac{\bar{X} - \bar{x}_{sys}}{\bar{X} + \bar{x}_{sys}} \right) \tag{7}$$

$$t_4 = \bar{y}_{sys} \exp\left(\frac{\bar{x}_{sys} - \bar{X}}{\bar{x}_{sys} + \bar{X}} \right) \tag{8}$$

$$Bias(t_3) = \lambda \bar{Y} \left(\frac{3}{8} B - \frac{1}{2} D \right) \tag{9}$$

$$Bias(t_4) = \lambda \bar{Y} \left(-\frac{1}{8} B + \frac{1}{2} D \right) \tag{10}$$

$$MSE(t_3) = \lambda \bar{Y}^2 (A + 0.25B - D) \tag{11}$$

$$MSE(t_4) = \lambda \bar{Y}^2 (A + 0.25B + D) \tag{12}$$

Both t_3 and t_4 give better estimate than t_0 if $\frac{B}{D} < 4$ and $\frac{B}{D} < -4$ respectively. Khan and Singh (2015) proposed chain ratio-type estimators of population mean under systematic random sampling by using two auxiliary variables. The proposed estimators and their mean squared errors are given below;

$$t_{\theta m} = \bar{y}_{sys}\left(\frac{\bar{X}}{\bar{X}+b_{yx}\left(\bar{x}_{sys}-\bar{X}\right)}\right)^{\delta_1}\left(\frac{\bar{Z}+b_{yz}\left(\bar{z}_{sys}-\bar{Z}\right)}{\bar{Z}}\right)^{\delta_2} \tag{13}$$

where δ_1 and δ_2 are unknowns to be estimated to minimized MSE of t_{m-}

$$MSE\left(t_{\theta m}\right) = \lambda\bar{Y}^2\left(A+\delta_1^2 b_{yx}C+\delta_1^2 b_{yx}B-2\delta_1 b_{yx}D+2\delta_2 b_{yZ}E-2\delta_1\delta_2 b_{yx}b_{yz}F\right) \tag{14}$$

Where $B = C_x^2\rho_x^*$, $E = \rho_{yz}\, C_y C_z\sqrt{\rho_y^*\rho_z^*}$,

The convectional ratio and product estimators for two-phase sampling and their biases and MSEs are given as follows;

$$t_{\theta 1} = \bar{y}_{sys}\frac{\bar{x}'_{sys}}{\bar{x}_{sys}} \tag{15}$$

$$t_{\theta 2} = \bar{y}_{sys}\frac{\bar{x}_{sys}}{\bar{x}'_{sys}} \tag{16}$$

$$B\left(t_{\theta 1}\right)_I = -\bar{Y}\left(\lambda_3\rho_x^*C_x^2+\frac{1}{2}\lambda_3\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{17}$$

$$B\left(t_{\theta 1}\right)_{II} = \bar{Y}\left(\lambda_1\rho_x^*C_x^2-\lambda_2\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{18}$$

$$B\left(t_{\theta 2}\right)_I = \bar{Y}\lambda_3\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*} \tag{19}$$

$$B\left(t_{\theta 2}\right)_{II} = \bar{Y}\left(\lambda_1\rho_x^*C_x^2+\lambda_2\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{20}$$

$$MSE\left(t_{\theta 1}\right)_I = \bar{Y}^2\left(\lambda_2\rho_y^*C_y^2+\lambda_3\rho_x^*C_x^2-2\lambda_3\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{21}$$

$$MSE\left(t_{\theta 1}\right)_{II} = \bar{Y}^2\left(\lambda_2\rho_y^*C_y^2+\lambda_3\rho_x^*C_x^2-2\lambda_2\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{22}$$

$$MSE\left(t_{\theta 2}\right)_I = \bar{Y}^2\left(\lambda_2\rho_y^*C_y^2+\lambda_3\rho_x^*C_x^2+2\lambda_3\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{23}$$

$$MSE\left(t_{\theta 2}\right)_{II} = \bar{Y}^2\left(\lambda_2\rho_y^*C_y^2+\lambda_3\rho_x^*C_x^2+2\lambda_2\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{24}$$

Singh et al. (2011) proposed estimators of population mean under two-phase sampling using systematic random sampling. The proposed estimators, their biases and MSEs are given below;

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$$t_{\theta 3}=\hat{t}_{sys}\exp\left(\frac{\bar{x}'_{sys}-\bar{x}_{sys}}{\bar{x}'_{sys}+\bar{x}_{sys}}\right) \tag{25}$$

$$t_{\theta 4}=\hat{t}_{sys}\exp\left(\frac{\bar{x}_{sys}-\bar{x}'_{sys}}{\bar{x}_{sys}+\bar{x}'_{sys}}\right) \tag{26}$$

$$B\left(t_{\theta 3}\right)_I=\bar{Y}\left(\frac{3}{8}\lambda_2\rho_x^*C_x^2-\frac{5}{8}\lambda_1\rho_x^*C_x^2-\frac{1}{2}\lambda_3\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{27}$$

$$B\left(t_{\theta 3}\right)_{II}=\bar{Y}\left(\frac{1}{8}\lambda_1\rho_x^*C_x^2+\frac{3}{8}\lambda_2\rho_x^*C_x^2-\frac{1}{2}\lambda_2\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{28}$$

$$B\left(t_{\theta 4}\right)_I=\bar{Y}\left(-\frac{1}{8}\lambda_3\rho_x^*C_x^2+\frac{1}{2}\lambda_3\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{29}$$

$$B\left(t_{\theta 4}\right)_{II}=\bar{Y}\left(-\frac{1}{8}\lambda_2\rho_x^*C_x^2+\frac{3}{8}\lambda_1\rho_x^*C_x^2+\frac{1}{2}\lambda_2\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{30}$$

$$MSE\left(t_{\theta 3}\right)_I=\bar{Y}^2\left(\lambda_2\rho_y^*C_y^2+\frac{1}{4}\lambda_3\rho_x^*C_x^2-\lambda_3\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{31}$$

$$MSE\left(t_{\theta 3}\right)_{II}=\bar{Y}^2\left(\lambda_2\rho_y^*C_y^2+\frac{1}{4}\lambda_3\rho_x^*C_x^2-\lambda_2\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{32}$$

$$MSE\left(t_{\theta 4}\right)_I=\bar{Y}^2\left(\lambda_2\rho_y^*C_y^2+\frac{1}{4}\lambda_3\rho_x^*C_x^2+\lambda_3\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{33}$$

$$MSE\left(t_{\theta 4}\right)_{II}=\bar{Y}^2\left(\lambda_2\rho_y^*C_y^2+\frac{1}{4}\lambda_3\rho_x^*C_x^2+\lambda_2\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}\right) \tag{34}$$

Abioye et al. (2019) suggested the following estimators of finite population mean under systematic sampling scheme;

$$t_{\theta 5}=\frac{\bar{y}_{sys}}{\bar{x}_{sys}}\left(\bar{x}'_{sys}+n'b_{xz}(\bar{z}'_{sys}-\bar{z}_{sys})\right) \tag{35}$$

$$t_{\theta 6}=\bar{y}_{sys}\exp\left(\frac{\bar{x}'_{sys}+n'b_{xz}(\bar{z}'_{sys}-\bar{z}_{sys})-\bar{x}_{sys}}{\bar{x}'_{sys}+n'b_{xz}(\bar{z}'_{sys}-\bar{z}_{sys})+\bar{x}_{sys}}\right) \tag{36}$$

$$B\left(t_{\theta 5}\right)_I=\bar{Y}\left[\lambda_2B-\lambda_2D+Rb_{xz}n'\bar{Z}\left(\lambda_3FF-\lambda_3E\right)\right] \tag{37}$$

$$B\left(t_{\theta 5}\right)_{II}=\bar{Y}\left[\lambda_2 B-\lambda_2 D+Rb_{xz}n'\bar{Z}\left(\lambda_2 FF-\lambda_2 E\right)\right] \tag{38}$$

$$B\left(t_{\theta 6}\right)_{I}=\bar{Y}\left[\frac{3}{4}\lambda_1 B\left(\frac{1}{2}\lambda_2-\lambda_1\right)-\frac{1}{8}\frac{n'^2b_{xz}^2\bar{Z}^2}{\bar{X}}\lambda_3 C-\lambda_3\frac{n'b_{xz}\bar{Z}}{4\bar{X}}FF-\lambda_3 D-\left(\lambda_3\frac{n'b_{xz}\bar{Z}}{2\bar{X}}E\right)\right] \tag{39}$$

$$B\left(t_{\theta 6}\right)_{II}=\bar{Y}\left[-\frac{1}{4}\lambda_1 B+\frac{3}{8}\lambda_2 B-\frac{1}{8}\frac{n'^2b_{xz}^2\bar{Z}^2}{\bar{X}}\lambda_4 C-\lambda_2\frac{n'b_{xz}\bar{Z}}{4\bar{X}}FF+\lambda_2 D-\left(\lambda_2\frac{n'b_{xz}\bar{Z}}{2\bar{X}}E\right)\right] \tag{40}$$

$$MSE\left(t_{\theta 5}\right)_{I}=\bar{Y}^2\left[\lambda_2 A+\lambda_3 B-2\lambda_3 D+\lambda_2 R^2 b_{xz}^2 n'^2\bar{Z}^2-2\lambda_3 n'\overline{YZ}Rb_{xz}\left(E-FF\right)\right] \tag{41}$$

$$MSE\left(t_{\theta 5}\right)_{II}=\bar{Y}^2\left[\lambda_2 A+\lambda_3 B-2\lambda_2 D\right]+\lambda_4 R^2 b_{xz}^2 n'^2\bar{Z}^2 C-2n'\overline{YZ}Rb_{xz}\left(\lambda_2 E+\lambda_4 FF\right) \tag{42}$$

$$MSE\left(t_{\theta 6}\right)_{I}=\bar{Y}^2\left[\lambda_2 A+\frac{\lambda_3}{4}B+\frac{n'b_{xz}^2\bar{Z}^2}{4\bar{X}}\lambda_3 C+\lambda_3\frac{n'b_{xz}\bar{Z}}{4\bar{X}^2}D-\lambda_3\frac{n'b_{xz}\bar{Z}}{\bar{X}}E+\lambda_3\frac{n'b_{xz}\bar{Z}}{2\bar{X}}FF\right] \tag{43}$$

$$MSE\left(t_{\theta 6}\right)_{I}=\bar{Y}^2\left[\lambda_2 A+\frac{\lambda_4}{4}B+\frac{n'b_{xz}^2\bar{Z}^2}{4\bar{X}^2}\lambda_4 C-\lambda_2 D-\lambda_2\frac{n'b_{xz}\bar{Z}}{\bar{X}}E+\lambda_1\frac{n'b_{xz}\bar{Z}}{2\bar{X}}B+\lambda_2\frac{n'b_{xz}\bar{Z}}{2\bar{X}}FF\right] \tag{44}$$

where

$$A=C_y^2\rho_y^*,\quad D=\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*},\quad E=\rho_{yz}C_yC_z\sqrt{\rho_y^*\rho_z^*},\quad B=C_x^2\rho_x^*,$$
$$C=C_z^2\rho_z^*,\quad FF=\rho_{xz}C_xC_z\sqrt{\rho_x^*\rho_z^*}$$

2.1 Proposed Estimator

Having studied the work of Singh et al. (2011), Khan and Singh (2015) and motivated by the work of Abioye et al. (2019), the following estimator of finite population mean in systematic random sampling was proposed as;

$$t_m^{(d)}=\bar{y}_{sys}\left(\frac{\bar{x}'_{sys}}{\bar{x}'_{sys}+b_{yx}\left(\bar{x}_{sys}-\bar{x}'_{sys}\right)}\right)^{\alpha_1}\left(\frac{\bar{z}'_{sys}+b_{yz}\left(\bar{z}_{sys}-\bar{z}'_{sys}\right)}{\bar{z}'_{sys}}\right)^{\alpha_2} \tag{45}$$

Where α_1 and α_2 are unknown functions to be estimated to minimized the MSE of $t_m^{(d)}$ .To obtain the MSE of the proposed estimator $t_m^{(d)}$, we use first order approximation of Taylor series method and the following error terms are defined;

$$\varepsilon_0=\frac{\left(\bar{y}_{sys}-\bar{Y}\right)}{\bar{Y}},\varepsilon_1=\frac{\left(\bar{x}_{sys}-\bar{X}\right)}{\bar{X}},\varepsilon_1'=\frac{\left(\bar{x}_1'-\bar{X}\right)}{\bar{X}},\varepsilon_2=\frac{\left(\bar{z}_{sys}-\bar{Z}\right)}{\bar{Z}},\varepsilon_2'=\frac{\left(\bar{z}'_{sys}-\bar{Z}\right)}{\bar{Z}}$$

such that

$$\left|\varepsilon_0\right|<1,\left|\varepsilon_1\right|<1,\left|\varepsilon_1'\right|<1,\left|\varepsilon_2\right|<1,\left|\varepsilon_2'\right|<1.$$

The expectation results of the error term are:

Under case I:

$$\left.\begin{array}{l}
E\left(\varepsilon_0\right)=E\left(\varepsilon_1\right)=E\left(\varepsilon_1^{'}\right)=E\left(\varepsilon_2\right)=E\left(\varepsilon_2^{'}\right)=0, E\left(\varepsilon_0^2\right)=\lambda_2 A, E\left(\varepsilon_1^2\right)=\lambda_2 B,\\
E\left(\varepsilon_1^{'2}\right)=\lambda_1 B, E\left(\varepsilon_2^2\right)=\lambda_2 C, E\left(\varepsilon_2^{'2}\right)=\lambda_1 C, E\left(\varepsilon_0\varepsilon_1\right)=\lambda_2 D, E\left(\varepsilon_0\varepsilon_1^{'}\right)=\lambda_1 D,\\
E\left(\varepsilon_0\varepsilon_2\right)=\lambda_2 E, E\left(\varepsilon_0\varepsilon_2^{'}\right)=\lambda_1 E, E\left(\varepsilon_1\varepsilon_1^{'}\right)=\lambda_1 B, E\left(\varepsilon_1\varepsilon_2\right)=\lambda_2 F, E\left(\varepsilon_1\varepsilon_2^{'}\right)=\lambda_1 F,\\
E\left(\varepsilon_2\varepsilon_1^{'}\right)=\lambda_1 F, E\left(\varepsilon_1^{'}\varepsilon_2^{'}\right)=\lambda_1 F, E\left(\varepsilon_2\varepsilon_2^{'}\right)=\lambda_1 C\ , \lambda_2=\dfrac{1}{n}-\dfrac{1}{N}, \lambda_1=\dfrac{1}{n'}-\dfrac{1}{N}, A=C_y^2\rho_y^*,\\
B=C_x^2\rho_x^*, C=C_z^2\rho_z^*, D=\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}, E=\rho_{yz}C_yC_z\sqrt{\rho_y^*\rho_z^*}, FF=\rho_{xz}C_xC_z\sqrt{\rho_x^*\rho_z^*}
\end{array}\right\} \tag{46}$$

Under case II:

$$\left.\begin{array}{l}
E\left(\varepsilon_0\right)=E\left(\varepsilon_1\right)=E\left(\varepsilon_1^{'}\right)=E\left(\varepsilon_2\right)=E\left(\varepsilon_2^{'}\right)=0, E\left(\varepsilon_0^2\right)=\lambda_2 A, E\left(\varepsilon_1^2\right)=\lambda_2 B,\\
E\left(\varepsilon_1^{'2}\right)=\lambda_1 B, E\left(\varepsilon_2^2\right)=\lambda_2 C, E\left(\varepsilon_2^{'2}\right)=\lambda_1 C, E\left(\varepsilon_0\varepsilon_1\right)=\lambda_2 D, E\left(\varepsilon_1^{'}\varepsilon_2^{'}\right)=\lambda_1 F\\
E\left(\varepsilon_0\varepsilon_2\right)=\lambda_2 E,\ E\left(\varepsilon_1\varepsilon_2\right)=\lambda_2 F, \lambda=\dfrac{1}{n}-\dfrac{1}{N}, A=C_y^2\rho_y^*,\ B=C_x^2\rho_x^*,\\
C=C_z^2\rho_z^*, D=\rho_{yx}C_yC_x\sqrt{\rho_y^*\rho_x^*}, E=\rho_{yz}C_yC_z\sqrt{\rho_y^*\rho_z^*}, FF=\rho_{xz}C_xC_z\sqrt{\rho_x^*\rho_z^*}\\
E\left(\varepsilon_0\varepsilon_1^{'}\right)=E\left(\varepsilon_2\varepsilon_1^{'}\right)=E\left(\varepsilon_2\varepsilon_2^{'}\right)=E\left(\varepsilon_0\varepsilon_2^{'}\right)=E\left(\varepsilon_1\varepsilon_1^{'}\right)=E\left(\varepsilon_1\varepsilon_2^{'}\right)=0,
\end{array}\right\} \tag{47}$$

Expressing (45) in terms of error terms, we have

$$t_m^{(d)}=\bar{Y}\left(1+\varepsilon_0\right)\left(\frac{\left(1+\varepsilon_1^{'}\right)}{\left(1+\varepsilon_1^{'}\right)+b_{yx}\left(\left(1+\varepsilon_1\right)-\left(1+\varepsilon_1^{'}\right)\right)}\right)^{\alpha_1}\left(\frac{\left(1+\varepsilon_2^{'}\right)+b_{yz}\left(\left(1+\varepsilon_2\right)-\left(1+\varepsilon_2^{'}\right)\right)}{\left(1+\varepsilon_2^{'}\right)}\right)^{\alpha_2} \tag{48}$$

$$t_m^{(d)}=\bar{Y}\left(1+\varepsilon_0\right)\left(1+b_{yx}\left(1+\varepsilon_1^{'}\right)^{-1}\left(\varepsilon_1-\varepsilon_1^{'}\right)\right)^{-\alpha_1}\left(1+b_{yz}\left(1-\varepsilon_2^{'}+\varepsilon_2^{'2}\right)^{-1}\left(\varepsilon_2-\varepsilon_2^{'}\right)\right)^{\alpha_2} \tag{49}$$

Simplify (49) up to first order approximation, we have

$$t_m^{(d)} = \bar{Y}\Big(1-\alpha_1 b_{yx}\left(\varepsilon_1-\varepsilon_1^{'}-\varepsilon_1\varepsilon_1^{'}+\varepsilon_1^{'2}\right)+\alpha_1\left(\alpha_1+1\right)/2\left(\varepsilon_1-\varepsilon_1^{'}\right)^2+\varepsilon_0-\alpha_1 b_{yx}\left(\varepsilon_0\varepsilon_1-\varepsilon_0\varepsilon_1^{'}\right)$$
$$+\alpha_2 b_{yz}\left(\varepsilon_2-\varepsilon_2^{'}-\varepsilon_2\varepsilon_2^{'}+\varepsilon_2^{'2}\right)+\alpha_2\left(\alpha_2-1\right)/2\left(\varepsilon_2-\varepsilon_2^{'}\right)^2+\alpha_2 b_{yz}\left(\varepsilon_0\varepsilon_2-\varepsilon_0\varepsilon_2^{'}\right)\Big) \quad (50)$$

Subtract from both sides of (50), we have

$$t_m^{(d)}-\bar{Y} = \bar{Y}\Big(\varepsilon_0-\alpha_1 b_{yx}\left(\varepsilon_1-\varepsilon_1^{'}-\varepsilon_1\varepsilon_1^{'}+\varepsilon_1^{'2}\right)+\alpha_1\left(\alpha_1+1\right)/2\left(\varepsilon_1-\varepsilon_1^{'}\right)^2-\alpha_1 b_{yx}\left(\varepsilon_0\varepsilon_1-\varepsilon_0\varepsilon_1^{'}\right)$$
$$+\alpha_2 b_{yz}\left(\varepsilon_2-\varepsilon_2^{'}-\varepsilon_2\varepsilon_2^{'}+\varepsilon_2^{'2}\right)+\alpha_2\left(\alpha_2-1\right)/2\left(\varepsilon_2-\varepsilon_2^{'}\right)^2+\alpha_2 b_{yz}\left(\varepsilon_0\varepsilon_2-\varepsilon_0\varepsilon_2^{'}\right)\Big) \quad (51)$$

By taking expectation of (51) and apply the results of (46) and (47), we obtain the bias of the proposed estimator $t_m^{(d)}$ under cases I and II as

$$Bias\left(t_m^{(d)}\right)_I = \bar{Y}\lambda_3\left(-\alpha_1 b_{yx}D+\frac{\alpha_1\left(\alpha_1+1\right)}{2}B+\alpha_2 b_{yz}E+\frac{\alpha_2\left(\alpha_2-1\right)}{2}C-\alpha_1\alpha_2 b_{yx}b_{yz}FF\right) \quad (52)$$

$$Bias\left(t_m^{(d)}\right)_{II} = \bar{Y}\Big(-\alpha_1 b_{yx}\left(\lambda_1 B+\lambda_2 D\right)+\frac{\alpha_1\left(\alpha_1+1\right)}{2}\lambda_4 B$$
$$+\alpha_2 b_{yz}\left(\lambda_1 C+\lambda_2 E\right)+\frac{\alpha_2\left(\alpha_2-1\right)}{2}\lambda_4 C-\alpha_1\alpha_2 b_{yx}b_{yz}\lambda_4 FF\Big) \quad (53)$$

Also, by taking expectation of (51), square the results and apply the results of (46) and (47), we obtain the MSE of the proposed estimator $t_m^{(d)}$ under cases I and II a

$$MSE\left(t_m^{(d)}\right)_I = \bar{Y}^2\left(\lambda_2 A+\lambda_3\left(\alpha_1^2 b_{yx}^2 B+\alpha_2^2 b_{yz}^2 C-2\left(\alpha_1 b_{yx}D-\alpha_2 b_{yz}E+\alpha_1\alpha_2 b_{yx}b_{yz}FF\right)\right)\right) \quad (54)$$

$$MSE\left(t_m^{(d)}\right)_{II} = \bar{Y}^2\left(\lambda_2 A+\lambda_4\left(\alpha_1^2 b_{yx}^2 B+\alpha_2^2 b_{yz}^2 C-2\alpha_1\alpha_2 b_{yx}b_{yz}FF\right)-2\lambda_2\left(\alpha_1 b_{yx}D-\alpha_2 b_{yz}E\right)\right) \quad (55)$$

To obtain the expression for function α_1 and α_2 for (54), we differentiate (54) partially with respect to α_1 and α_2 and equate them to zero as;

$$\frac{\partial MSE\left(t_m^{(d)}\right)_I}{\partial\alpha_1} = \alpha_1 b_{yx}B-D-\alpha_2 b_{yz}FF=0 \quad (56)$$

$$\alpha_1 = \frac{D+\alpha_2 b_{yz}FF}{b_{yx}B} \quad (57)$$

$$\frac{\partial MSE\left(t_m^{(d)}\right)_I}{\partial\alpha_2} = \alpha_2 b_{yz}C+E-\alpha_1 b_{yx}FF=0 \quad (58)$$

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$$\alpha_2 = \frac{\alpha_1 b_{yx} FF - E}{b_{yz} C} \tag{59}$$

Solve (57) and (59) simultaneously; we obtain as the expression for optimal α_1 and α_2 denoted by α_1^{opt} and α_2^{opt} respectively as

$$\left(\alpha_1^{opt}\right)_I = \frac{CD - EFF}{b_{yx}\left(BC - FF^2\right)} \tag{60}$$

$$\left(\alpha_2^{opt}\right)_I = \frac{DFF - BE}{b_{yz}\left(BC - FF^2\right)} \tag{61}$$

Substitute (60) and (61) in (54), we obtain minimum MSE of the proposed estimator under case I.

To obtain the expression for function α_1 and α_2 for (55), we differentiate (55) partially with respect to α_1 and α_2 and equate them to zero as;

$$\frac{\partial MSE\left(t_m^{(d)}\right)_{II}}{\partial \alpha_1} = \lambda_4 \alpha_1 b_{yx} B - \lambda_2 D - \lambda_4 \alpha_2 b_{yz} FF = 0 \tag{62}$$

$$\alpha_1 = \frac{\alpha_2 \lambda_4 b_{yz} FF + \lambda_2 D}{\lambda_4 b_{yx} B} \tag{63}$$

$$\frac{\partial MSE\left(t_m^{(d)}\right)_{II}}{\partial \alpha_2} = \alpha_2 \lambda_4 b_{yz} C + \lambda_2 E - \alpha_1 \lambda_4 b_{yx} FF = 0 \tag{64}$$

$$\alpha_2 = \frac{\alpha_1 \lambda_4 b_{yx} FF - \lambda_2 E}{\lambda_4 b_{yz} C} \tag{65}$$

Solve (63) and (65) simultaneously; we obtain as the expression for optimal α_1 and α_2 denoted by α_1^{opt} and α_2^{opt} respectively as

$$\left(\alpha_1^{opt}\right)_{II} = \frac{\lambda_2\left(CD - EFF\right)}{\lambda_4 b_{yx}\left(BC - FF^2\right)} \tag{66}$$

$$\left(\alpha_2^{opt}\right)_{II} = \frac{\lambda_2\left(DF - BE\right)}{\lambda_4 b_{yz}\left(BC - F^2\right)} \tag{67}$$

Substitute (66) and (67) in (55), we obtain minimum MSE of the proposed estimator under case II.

3.0 Empirical Study

In this section, empirical study is conducted to compare the efficiency of the proposed estimator and that of existing estimators considered in section two. The following data were used for the empirical study

.Data 1: Tailor *et al.* (2013)

$N = 15, n = 3, \acute{n} = 21. \bar{Y} = 80, , \bar{X} = 44.47, \bar{Z} = 48.40, C_y = 0.56, C_x = 0.28$

$C_z = 0.43, S_y^2 = 2000, S_x^2 = 149.55, S_Z^2 = 427.83, S_{YX} = 538.57, S_{yz} = -902.86$

$S_{xz} = -241.06, \rho_{yx} = 0.9848, \rho_{yz} = -0.9760, \rho_{xz} = -0.9530,$

$\rho_y = 0.6652, \rho_x = 0.707, \rho_z = 0.5487$

Table 1. The MSE and PRE of the Proposed and Existing Estimators

Estimators	MSE	PRE
CASE I		
t_0 .	13096.18	100
$t_{(\theta 1)_I}$.	2854.343	458.8159
$t_{(\theta 3)_I}$.	1210.346	1082.019
$t_{(\theta 5)_I}$.	2878099	0.4550288
$t_{(\theta 6)_I}$.	494453.3	2.648618
$\left(t_m^{(d)}\right)_I$.	1000.683	1308.724
CASE II		
t_0 .	13096.18	100
$t_{(\theta 1)_{II}}$.	2935.486	446.1331
$t_{(\theta 3)_{II}}$.	1250.918	1046.925
$t_{(\theta 5)_{II}}$.	763043.3	1.716308
$t_{(\theta 6)_{II}}$.	4607.814	284.2167
$\left(t_m^{(d)}\right)_{II}$.	934.687	1401.13

Data 2: Tailor *et al.* (2013)

$N = 200,\ \ n = 7,\ n' = 25,\ \bar{Y} = 80.1,\ \bar{X} = 44.47,\ \bar{Z} = 48.40,\ C_y = 0.56,\ C_x = 0.28,\ C_z = 0.43,\ S_y^2 = 2000,$
$S_x^2 = 149.55, S_z^2 = 427.83,\ S_{yx} = 538.57, S_{yz} = -902.86, S_{xz} = -241.06,\ \rho_{yx} = 0.5, \rho_{yz} = -0.5, \rho_{xz} = -0.5,$
$\rho_y = 0.6652, \rho_x = 0.707, \rho_z = 0.5487.$

Table 2. The MSE and PRE of the Proposed and Existing Estimators

Estimators	**MSE**	**PRE**
CASE I		
t_0 .	69946.69	100
$t_{(\theta 1)_I}$.	1476.67	4736.786
$t_{(\theta 3)_I}$.	1320.11	5298.549
$t_{(\theta 5)_I}$.	59697.48	117.1686
$t_{(\theta 6)_I}$.	14909.21	469.1509
$\left(t_m^{(d)}\right)_I$.	1290.25	5421.172
CASE II		
t_0 .	69946.69	100
$t_{(\theta 1)_{II}}$.	1452.523	4815.531
$t_{(\theta 3)_{II}}$.	1308.037	5347.456
$t_{(\theta 5)_{II}}$.	51910.31	134.7453
$t_{(\theta 6)_{II}}$.	1310.117	5338.965
$\left(t_m^{(d)}\right)_{II}$.	1288.628	5427.995

Data 3: Anderson, (1958) : Weight (kg) of the children, X : Head length of first son, Z : Head breadth of the first son.

$N = 25,\ \ n = 7,\ n' = 10,\ \bar{Y} = 183.34,\ \bar{X} = 185.72,\ \bar{Z} = 151.12,\ C_y = 0.0546,\ C_x = 0.2422,\ C_z = 0.0488,$
$\rho_{yx} = 0.7108, \rho_{yz} = 0.69320, \rho_{xz} = 0.73460, \beta_1(z) = 0.002, \beta_2(z) = 2.6519.$

Table 3.The MSE and PRE of the Proposed and Existing Estimators

Estimators	**MSE**	**PRE**

CASE I		
t_0 .	3361.042	100
$t_{(\theta 1)_I}$.	447.64	750.836
$t_{(\theta 3)_I}$.	143.2622	2346.078
$t_{(\theta 5)_I}$.	4596680	0.07311891
$t_{(\theta 6)_I}$.	81569.38	4.120471
$\left(t_m^{(d)}\right)_I$.	47.99372	7003.088
CASE II		
t_0 .	3361.042	100
$t_{(\theta 1)_{II}}$.	421.6387	797.1381
$t_{(\theta 3)_{II}}$.	130.2615	2580.226
$t_{(\theta 5)_{II}}$.	64540.32	5.207663
$t_{(\theta 6)_{II}}$.	12960.65	25.93266
$\left(t_m^{(d)}\right)_{II}$.	47.99372	7003.087

Data 4: Singh and Kumar, (2011) : Weight (kg) of the children,X : Skull circumference (cm) of the children, Z : Chest circumference of the children.

Table 4.The MSE and PRE of the Proposed and Existing Estimators

Estimators	MSE	PRE

CASE I		
t_0 .	3586.673	100
$t_{(\theta 1)_I}$.	4.672474	76761.76
$t_{(\theta 3)_I}$.	4.620488	77625.43
$t_{(\theta 5)_I}$.	10.98113	32662.14
$t_{(\theta 6)_I}$.	6.850697	52354.87
$\left(t_m^{(d)}\right)_I$.	4.606655	77858.51
CASE II		
t_0 .	3586.673	100
$t_{(\theta 1)_{II}}$.	4.657991	77000.43
$t_{(\theta 3)_{II}}$.	4.613246	77747.28
$t_{(\theta 5)_{II}}$.	28.04281	12789.99
$t_{(\theta 6)_{II}}$.	4.951752	72432.41
$\left(t_m^{(d)}\right)_{II}$.	4.606662	77858.41

4.0 Conclusion

In this research paper, an estimator of finite population mean under systematic random sampling has been suggested. The MSE of the proposed estimator has been derived up to second degree approximation using Taylor series approach. In addition, numerical illustration on the efficiency of the proposed estimator in comparison to other estimators considered in the study using four real life data sets was done and the results revealed that the proposed estimator has least MSE and highest PRE. This implies that the proposed estimator is more efficient compared to others. In future, other improvement strategies such as linear combination, exponential e.t.c can be used to improve the proposed estimator.

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