Annals of Optimization Theory and Practice

Volume 4, Number 1, 87-102
April 2021

DOI: 10.22121/AOTP.2021.285741.1066

The new generalized averaging aggregation operators and their application on group decision making problem base on interval-valued Pythagorean fuzzy numbers

Khaista Rahman

Hazara University Mansehra, Pakistan
✉ khaista355@yahoo.com

(Received: February 27, 2021 / Accepted: April 27, 2021)

Abstract The objective of this paper is to present the concept of the Interval-valued Pythagorean fuzzy set (IVPFS). Interval-valued Pythagorean fuzzy set is one of the successful extensions of the interval-valued intuitionistic fuzzy set. Under this environment, in this paper we introduced the notion of some generalized aggregation operation such as generalized interval-valued Pythagorean fuzzy weighted averaging (GIVPFWA) operator, generalized interval-valued Pythagorean fuzzy ordered weighted averaging (GIVPFOWA) operator, generalized interval-valued Pythagorean fuzzy hybrid averaging (GIVPFHA) operator long with their desirable properties namely, idempotency, boundedness and monotonicity. The main advantage and merits of using the proposed operators is that these operators give a complete view of the problem to the decision makers. These methods provide more general, more accurate and precise results as compared to the existing methods. Therefore these methods play a vital role in real world problems. Finally the proposed operators have been applied to decision making problems to show the validity and effectiveness of the new approach.

Keywords GIVPFWA operator; GIVPFOWA operator; GIVPFHA operator

1. Introduction

Multi-criteria decision making (MCDM) problems are of importance in most kinds of fields such as engineering, economics, and management. Traditionally, it has been assumed that the information which accesses the alternatives in term of criteria and weight are expressed in real numbers. But due to the complexity of the system day-by-day, it is difficult for the decision makers to make a perfect decision, as most of the preferred value during the decision-making (DM) process imbued with uncertainty. In order to handle the uncertainties, intuitionistic fuzzy set(IFS) Atanassov,(1986) theory is

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one of the successful extension of the fuzzy set (FS) theory Zadeh,(1965), which is characterized by the degree of membership and degree of non-membership has been presented. Later on, Atanassov and Gargov, (1989) extended it to the interval-valued intuitionistic fuzzy sets (IVIFS), which is characterized by a membership degree and a non-membership degree, whose values are intervals rather than real numbers. Over the last four decades, the IFS and IVIFS have received more and more attention by introducing the various kinds of aggregation operators, information measures and employed them to solve the decision-making problems under the different environment (Garg,(2016);Garg,(2016); Su *et al*.,(2011); Kumar and Garg,(2016); Wei and Wang,(2007); Xu and Jain ,(2007); Wang *et al*.,(2009)). However, apart from these, Xu,(2010), Tan and Chen,(2010), Garg *et al*.,(2017), used the Choquet integral to develop some intuitionistic fuzzy aggregation operators, which not only consider the importance of the elements or their ordered positions, but also can reflect the correlations among the elements or their ordered positions. (Rahman *et al*.,(2018); Rahman *et al*.,(2020)) and Jamil *et al*.,(2020) introduced the idea of some generalized operators and applied them on decision making problem.

But the limitation of their studies is that they are valid only for those environments whose degrees sum is less than one. However, in day-to-day life, there are many situations where this condition is ruled out. For instance, if a person gives their preference in the form of membership and non-membership degrees towards a particular object is 0.8 and 0.6, and then clearly this situation is not handling with IFS. In order to resolve it, Yager (Yager,(2013); Yager,(2014)) proposed the Pythagorean fuzzy set (PFS) by relaxing this sum condition to its square sum less than one. For instance, corresponding to the above-considered example, we see that $(0.8)^2 + (0.6)^2 = 1$ and hence PFS is an extension of the existing IFS. After their pioneer work, Yager and Abbasov ,(2013) introduced the notion of the Pythagorean fuzzy weighted averaging (PFWA) operator and Pythagorean fuzzy ordered weighted averaging (PFOWA) operator. Zeng and Xu ,(2014) introduced the notion of TOPSIS method using Pythagorean fuzzy numbers. Peng and Yang, (2015) developed some important results for Pythagorean fuzzy sets. (Garg ,(2016); Garg ,(2017)) used the Einstein sum and Einstein product and introduced the notion of Pythagorean fuzzy Einstein arithmetic aggregation operators and Pythagorean fuzzy Einstein geometric aggregation operators such as, Pythagorean fuzzy Einstein weighted averaging (PFEWA) operator, Pythagorean fuzzy Einstein ordered weighted averaging (PFEOWA) operator, generalized Pythagorean fuzzy Einstein weighted averaging (GPFEWA) operator, generalized Pythagorean fuzzy Einstein ordered weighted averaging (GPFEOWA) operator, Pythagorean fuzzy Einstein weighted geometric (PFEWG) operator, Pythagorean fuzzy Einstein ordered weighted geometric (PFEOWG) operator, generalized Pythagorean fuzzy Einstein weighted geometric (GPFEWG) operator, generalized Pythagorean fuzzy Einstein ordered weighted geometric (GPFEOWG) operator and also applied them to group decision making. Garg,(2017), further, presented a confidence level based aggregation operators, by incorporating the confidence level of the decision makers to the analysis, and named as confidence based Pythagorean fuzzy weighted averaging and geometric aggregation operators.

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But, in some real decision-making problems, due to insufficiency in available information, it may be difficult for decision makers to exactly quantify their opinions with a crisp number, but they can be represented by an interval number within [0,1]. Therefore it is so important to present the idea of interval-valued Pythagorean fuzzy sets (IVPFSs), which permit the membership degrees and non- membership degrees to a given set to have an interval value. Zhang,(2016) introduced the concept of interval-valued Pythagorean fuzzy set. Peng and Yang ,(2015) introduced the notion of, interval-valued Pythagorean fuzzy weighted averaging (IVPFWA) operator, interval-valued Pythagorean fuzzy weighted geometric (IVPFWG) operator and also introduced some of their fundamental and important properties. Garg,(2016) presented an interval-valued Pythagorean fuzzy weighted average (IVPFWA) and interval-valued Pythagorean fuzzy weighted geometric (IVPFWG) operators for solving the decision-making problem under IVPFS environment. Also, a novel accuracy function has been defined in it for ranking the different interval-valued Pythagorean fuzzy numbers (IVPFNs). Now, in order to compare the interval numbers, some score, as well as accuracy function, have been taken for measurement and then applied to solve MCDM problems. Garg,(2016) defined the concepts of correlation and correlation coefficients of PFSs. Garg,(2017) also presented an improved accuracy under the IVPFS for solving the decision-making problems. (Rahman *et al*., (2017); Rahman *et al*., (2018); Rahman *et al*., (2019); Rahman *et al*., (2019); Rahman *et al*., (2019); Rahman *et al*., (2019); Rahman *et al*., (2020)) introduced the concept of several aggregations and induced aggregation operators and applied them to multiple attribute group decision making. (Garg *et al*., (2021); Sheng, (2021)) developed some new methods for decision making problems.Wang and Li, (2021) ntroduced the notion of Pythagorean fuzzy interaction power Bonferroni mean aggregation operator. Wang *et al*., (2020) introduced the idea of Pythagorean fuzzy interactive Hamacher power aggregation operators and applied them to multiple attribute group decision making. Shahzadi *et al*., (2020) developed some operators based on Pythagorean fuzzy numbers and applied them on decision making problems.

Thus keeping the advantages of the above mention aggregation operators, in this paper we introduce a series of new generalized interval-valued aggregation operators based on IVPFNs, such as, GIVPFWA operator, GIVPFOWA operator, GIVPFHA operators, and apply them to group decision making problems. We also discuss some of their basic properties including idempotency, boundedness, commutativity and monotonicity and give some examples to develop these proposed operators. These operators provide more accurate and precise results as compare to the existing methods. At the last of the paper we present an application of these proposed operators. Comparing with interval-valued Pythagorean fuzzy averaging aggregation operators proposed by Rahman *et al*.,(2018), they are only the special cases of the proposed operators in this paper. When $\delta = 1$, then the generalized interval-valued Pythagorean fuzzy weighted averaging aggregation operator, generalized interval-valued Pythagorean fuzzy ordered weighted averaging aggregation operator, generalized interval-valued Pythagorean fuzzy hybrid averaging aggregation operator proposed in this paper can be reduce to interval-valued Pythagorean fuzzy weighted averaging aggregation operator, interval-valued Pythagorean fuzzy ordered weighted averaging aggregation operator, interval-valued Pythagorean fuzzy hybrid averaging aggregation operator respectively. Obviously, the operators and methods proposed in this paper are more general. Of course, superficially, it is more complicated in calculation. However, in real applications, we need assign the specific parameter δ, firstly.

The remainder paper can be constructed as. In Section 2, we present some basic definition. In Section 3, we introduce the notion of GIVPFWA operator, GIVPFOWA operator and GIVPFHA operator. In Section 4, we develop an application of the proposed operators. In Section 5, we construct a numerical example. In Section 6, we have conclusion.

2. Preliminaries

Definition 1: (Peng andYang,(2015); Garg,(2016)) Let K be a universal set, then IVPFS, I in K can be defined as:

$$I = \{\langle k, \mu_I(k), \nu_I(k)\rangle | k \in K\} \tag{1}$$

Where

$$\mu_I(k) = \left[\mu_I^a(k), \mu_I^b(k)\right] \subset [0,1], \nu_I(k) = \left[\nu_I^a(k), \nu_I^b(k)\right] \subset [0,1],\ \mu_I^a(k) = \inf\left(\mu_I(k)\right)$$

$$\mu_I^b(k) = \sup\left(\mu_I(k)\right), \nu_I^a(k) = \inf\left(\nu_I(k)\right), \nu_I^b(k) = \sup\left(\nu_I(k)\right),$$

$$0 \le \left(\mu_I^b(k)\right)^2 + \left(\nu_I^b(k)\right)^2 \le 1 \text{ and } \pi_I(k) = \left[\pi_I^a(k), \pi_I^b(k)\right], \text{ for all } k \in K.$$

Definition 2: (Peng and Yang, (2015); Garg, (2016)) Let $\lambda = ([p_\lambda, q_\lambda], [r_\lambda, t_\lambda])$ be IVPFV, then the scores and accuracy of λ are $S(\lambda) = \frac{1}{2}[(p_\lambda)^2 + (q_\lambda)^2 - (r_\lambda)^2 - (t_\lambda)^2]$ and $H(\lambda) = \frac{1}{2}[(p_\lambda)^2 + (q_\lambda)^2 + (r_\lambda)^2 + (t_\lambda)^2]$ respectively.

Note:$S(\lambda_1) < S(\lambda_2)$

1) If $S(\lambda_1) < S(\lambda_2)$Then $(\lambda_1) < (\lambda_2$
2) If $S(\lambda_1) = S(\lambda_2)$ then

(i) If $H(\lambda_1) = H(\lambda_2)$ then $(\lambda_1) = (\lambda_2)$

(ii) If $H(\lambda_1) < H(\lambda_2)$ then $(\lambda_1) < (\lambda_2)$

(iii) If $H(\lambda_1) > H(\lambda_2)$ then $(\lambda_1) > (\lambda_2)$

Definition 3: Peng andYang, (2015)Let $\lambda_j = \left(\left[p_{\lambda_j}, q_{\lambda_j}\right], \left[r_{\lambda_j}, t_{\lambda_j}\right]\right) (j = 1,2, \dots, n)$ be a group of IVPFNs, $\delta > 0$, then

$$\delta\lambda_1 = \left(\left[\sqrt{1 - (1 - (p_{\lambda_1})^2)^\delta}, \sqrt{1 - (1 - (q_{\lambda_1})^2)^\delta}\right], \left[(r_{\lambda_1})^\delta, (t_{\lambda_1})^\delta\right]\right) \tag{2}$$

$$\lambda_1{}^{\delta} = \left(\left[(r_{\lambda_1})^{\delta}, (t_{\lambda_1})^{\delta}\right], \sqrt{1-(1-(r_{\lambda_1})^2)^{\delta}}, \sqrt{1-(1-(t_{\lambda_1})^2)^{\delta}}\right) \quad (3)$$

$$\lambda_1 \otimes \lambda_2 = \left(\left[p_{\lambda_1}, p_{\lambda_2}, q_{\lambda_1}, q_{\lambda_2}\right], \left[\frac{\sqrt{(r_{\lambda_1})^2+(r_{\lambda_2})^2-(r_{\lambda_1})^2(r_{\lambda_2})^2}}{\sqrt{(t_{\lambda_1})^2+(t_{\lambda_2})^2-(t_{\lambda_1})^2(t_{\lambda_2})^2}}\right]\right) \quad (4)$$

$$\lambda_1 \oplus \lambda_2 = \left(\left[\frac{\sqrt{(p_{\lambda_1})^2+(p_{\lambda_2})^2-(p_{\lambda_1})^2(p_{\lambda_2})^2}}{\sqrt{(q_{\lambda_1})^2+(q_{\lambda_2})^2-(q_{\lambda_1})^2(q_{\lambda_2})^2}}\right], \left[r_{\lambda_1}, r_{\lambda_2}, t_{\lambda_1}, t_{\lambda_2}\right]\right) \quad (5)$$

Definition 4: (Rahman *et al.*, (2018)) The IVPFWA operator can be defined as

$$IVPFWA_{\varpi}(\lambda_1, \lambda_2, \dots, \lambda_n) = \left(\left[\sqrt{1-\prod_{j=1}^{n}\left(1-\left(p_{\lambda_j}\right)^2\right)^{\varpi_j}}, \sqrt{1-\prod_{j=1}^{n}\left(1-\left(q_{\lambda_j}\right)^2\right)^{\varpi_j}}\right], \left[\prod_{j=1}^{n}(r_{\lambda_j})^{\varpi_j}, \prod_{j=1}^{n}\right.\right. \quad (6)$$

Where $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$ is the weighted vector of $\lambda_j (j = 1,2,\dots,n)$ and $\varpi_j \in [0,1], \sum_{j=1}^{n} \varpi_j = 1$.

Example1:

Let, $\lambda_1 = ([0.3,0.4],[0.5,0.7]), \lambda_2 = ([0.2,0.6],[0.3,0.6]), \lambda_3 = ([0.3,0.6],[0.3,0.5])$ and $\lambda_4 = ([0.4,0.7],[0.2,0.6])$ be the four IVPFVs and $\varpi = (0.1,0.2,0.3,0.4)^T$ be their weighted vector. Now applying the IVPEWA operator, we get the following result.

$$\text{IVPFWA}_{\varpi}(\lambda_1, \lambda_2, \lambda, \lambda_4) = \left(\left[\sqrt{1-\prod_{j=1}^{4}\left(1-p^2_{\lambda_j}\right)^{\varpi_j}}, \sqrt{1-\prod_{j=1}^{4}\left(1-q^2_{\lambda_j}\right)^{\varpi_j}}\right], \left[\prod_{j=1}^{4}\left(r_{\lambda_j}\right)^{\varpi_j}, \prod_{j=1}^{4}\left(t_{\lambda_j}\right)^{\varpi_j}\right]\right)$$

$$= ([0.330, 0.632], [0.268, 0.576])$$

Definition 5: (Rahman *et al.*, (2018)) The IVPFOWA operator can be defined as:

$$IVPFOWA_{\varpi}(\lambda_1, \lambda_2, \dots, \lambda_n) = \left(\left[\sqrt{1-\prod_{j=1}^{n}\left(1-\left(p_{\sigma_j}\right)^2\right)^{\varpi_j}}, \sqrt{1-\prod_{j=1}^{n}\left(1-\left(q_{\sigma_j}\right)^2\right)^{\varpi_j}}\right], \left[\prod_{j=1}^{n}(r_{\sigma_j})^{\varpi_j}, \prod_{j=1}^{n}(t_{\sigma_j})^{\varpi_j}\right]\right) \quad (7)$$

where $\lambda_{\sigma_{(j)}}$ is the jth largest value of $\lambda_{\sigma_{(j)}}$ and $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$ be the weighted vector.

Example2:

Let, $\lambda_1 = ([0.4,0.6],[0.3,0.7]), \lambda_2 = ([0.3,0.6],[0.2,0.7]), \lambda_3 = ([0.3,0.8],[0.3,0.5])$ and $\lambda_4 = ([0.4,0.9],[0.1,0.3])$ be the four IVPFVs and let $\varpi = (0.1,0.2,0.3,0.4)^T$ be the weighted vector. First we calculate the scores of $\lambda_j = (j = 1,2,3,4)$, thus we have $S(\lambda_1) = -0.03, S(\lambda_2) = -.004, S(\lambda_3) = 0.19, S(\lambda_4) = 0.43$ .Thus $S(\lambda_2) < S(\lambda_1) < S(\lambda_3) < S(\lambda_4)$

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Hence $\lambda_{\sigma(1)} = ([0.4,0.9],[0.1,0.3])$, $\lambda_{\sigma(2)} = ([0.3,0.8],[0.3,0.5])$, $\lambda_{\sigma(3)} = ([0.4,0.6],[0.3,0.7])$, and $\lambda_{\sigma(4)} = ([0.3,0.6],[0.2,0.7])$.

Now applying the IVPFOWA operator, we have

$$\mathrm{IVPFOWA}_{\varpi}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\left(\left[\sqrt{1-\prod_{j=1}^{4}\left(1-p^2_{\lambda_{\sigma(j)}}\right)^{\varpi_j}},\sqrt{1-\prod_{j=1}^{4}\left(1-q^2_{\lambda_{\sigma(j)}}\right)^{\varpi_j}}\right],\left[\prod_{j=1}^{4}\left(r_{\lambda_{\sigma(j)}}\right)^{\varpi_j},\prod_{j=1}^{4}\left(t_{\lambda_{\sigma(j)}}\right)^{\varpi_j}\right]\right)$$
$$=([0.344,0.703],[0.228,0.601])$$

Definition 6: (Rahman *et al.*, (2018)) The IVPFHA operator can be defined as:

$$IVPFHA_{\varpi}(\lambda_1,\lambda_2,\ldots,\lambda_n)$$
$$=\left(\left[\sqrt{1-\prod_{j=1}^{n}\left(1-\left(p_{\lambda_{\tau(j)}}\right)^2\right)^{\varpi_j}},\sqrt{1-\prod_{j=1}^{n}\left(1-\left(q_{\lambda_{\tau(j)}}\right)^2\right)^{\varpi_j}}\right],\left[\prod_{j=1}^{n}(r_{\lambda_{\sigma(j)}})^{\varpi_j},\prod_{j=1}^{n}(t_{\lambda_{\sigma(j)}})^{\varpi_j}\right]\right) \tag{8}$$

where $\varpi = (\varpi_1,\varpi_2,\ldots,\varpi_n)^T$ is the weighted vector of $\lambda_{\sigma(j)}$ and $\varpi \in [0,1], \sum_{j=1}^{n}\varpi_j = 1$.

Example3:

Let $\lambda_1 = ([0.4,0.7],[0.3,0.4])$, $\lambda_2 = ([0.3,0.6],[0.2,0.4])$, $\lambda_3 = ([0.3,0.7],[0.1,0.3])$ and $\lambda_4 = ([0.4,0.8],[0.1,0.3])$ and be the four IVPFVs whose weighted vector is $\varpi = (0.4,0.3,0.2,0.1)^T$.

First we calculate the hybrid values: $\lambda_1 = ([0.259,0.485],[0.617,0.693])$, $\lambda_2 = ([0.269,0.547],[0.257,0.480])$, $\lambda_3 = ([0.327,0.744],[0.235,0.435])$, $\lambda_4 = ([0.493,0.897],[0.025,0.145])$.

Now find the scores, we have

$$S\left(\dot{\lambda}_1\right)=-0.279, S\left(\dot{\lambda}_2\right)=0.032, S\left(\dot{\lambda}_3\right)=0.208, S\left(\dot{\lambda}_4\right)=0.513.$$

Hence we get the following: $\lambda_{\sigma(1)} = ([0.493,0.897],[0.025,0.145])$, $\lambda_{\sigma(2)} = ([0.327,0.744],[0.235,0.435])$, $\lambda_{\sigma(3)} = ([0.269,0.547],[0.275,0.480])$ and $\lambda_{\sigma(4)} = ([0.259,0.485],[0.617,0.693])$. Thus we get

$$\mathrm{IVPFHA}_{\varpi,\varpi}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$$
$$=\left(\left[\sqrt{1-\prod_{j=1}^{4}\left(1-p^2_{\dot{\lambda}_{\sigma(j)}}\right)^{\varpi_j}},\sqrt{1-\prod_{j=1}^{4}\left(1-q^2_{\dot{\lambda}_{\sigma(j)}}\right)^{\varpi_j}}\right],\left[\prod_{j=1}^{4}\left(r_{\dot{\lambda}_{\sigma(j)}}\right)^{\varpi_j},\prod_{j=1}^{4}\left(t_{\dot{\lambda}_{\sigma(j)}}\right)^{\varpi_j}\right]\right)$$
$$=([0.705,0.793],[0.109,0.300])$$

3. Some Generalized Aggregation Operators base on IVPFVs

In this section, we introduce the notion of GIVPFWA operator GIVPFOWA operator and GIVPFHA operator. We also discuss some desirable properties of these propose operators such as, idempotency, boundedness, commutatively, monotonicity.

Definition 7: The GIVPFWA operator can be defined as:

$$\mathrm{GIVPFWA}_{\varpi}\left(\tilde{\lambda}_1,\tilde{\lambda}_2,\ldots,\tilde{\lambda}_n\right)=\left(\begin{array}{c}\left[\left(\sqrt{1-\prod_{j=1}^{n}\left(1-p_{\tilde{\lambda}_j}^{2\delta}\right)^{\varpi_j}}\right)^{\frac{1}{\delta}},\left(\sqrt{1-\prod_{j=1}^{n}\left(1-q_{\tilde{\lambda}_j}^{2\delta}\right)^{\varpi_j}}\right)^{\frac{1}{\delta}}\right]\\ \left[\sqrt{1-\left[1-\prod_{j=1}^{n}\left(1-\left(1-r_{\tilde{\lambda}_j}^{2}\right)^{\delta}\right)^{\varpi_j}\right]^{\frac{1}{\delta}}},\sqrt{1-\left[1-\prod_{j=1}^{n}\left(1-\left(1-t_{\tilde{\lambda}_j}^{2}\right)^{\delta}\right)^{\varpi_j}\right]^{\frac{1}{\delta}}}\right]\end{array}\right) \tag{9}$$

where $\varpi=(\varpi_1,\varpi_2,\ldots,\varpi_n)^T$ be the weighted vector of with some conditions $\varpi\in[0,1]$ and $\sum_{j=1}^{n}\varpi_j=1$, and $\delta>0$.

Example4:

Let, $\lambda_1=([0.3,0.5],[0.4,0.8]),\lambda_2=([0.2,0.6],[0.3,0.7]),\lambda_3=([0.3,0.7],[0.2,0.5])$, and $\lambda_4=([0.4,0.5],[0.4,0.6])$ are four IVPFVs and let $\varpi=(0.1,0.2,0.3,0.4)^T$ and $\delta=2$, then we have the following result:

$$\mathrm{GIVPFWA}_{\varpi}\left(\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}_3,\tilde{\lambda}_4\right)=\left(\begin{array}{c}\left[\left(\sqrt{1-\prod_{j=1}^{4}\left(1-p_{\tilde{\lambda}_j}^{2\delta}\right)^{\varpi_j}}\right)^{\frac{1}{\delta}},\left(\sqrt{1-\prod_{j=1}^{4}\left(1-q_{\tilde{\lambda}_j}^{2\delta}\right)^{\varpi_j}}\right)^{\frac{1}{\delta}}\right]\\ \left[\sqrt{1-\left[1-\prod_{j=1}^{4}\left(1-\left(1-r_{\tilde{\lambda}_j}^{2}\right)^{\delta}\right)^{\varpi_j}\right]^{\frac{1}{\delta}}},\sqrt{1-\left[1-\prod_{j=1}^{4}\left(1-\left(1-t_{\tilde{\lambda}_j}^{2}\right)^{\delta}\right)^{\varpi_j}\right]^{\frac{1}{\delta}}}\right]\end{array}\right)$$

$$=\left[(0.343,0.603),\ (0.195,0.602)\right]$$

Theorem1: Let $\lambda_j=(\left[p_{\lambda_j},q_{\lambda_j}\right],\left[r_{\lambda_j},t_{\lambda_j}\right])$ be the collection IVPFVs, and $\varpi=(\varpi_1,\varpi_2,\ldots,\varpi_n)^T$be the weighted vector of $\lambda_j(j=1,2,\ldots,n)$ with conditions, $\varpi_J\in[0,1]$ and $\sum_{j=1}^{n}\varpi_j=1$, then

i) **Commutativity**: If $\lambda_j{}^*=(\left[p_{\lambda_j{}^*},q_{\lambda_j{}^*}\right],\left[r_{\lambda_j{}^*},t_{\lambda_j{}^*}\right])$ be a set of IVPFNs, then

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$$GIVPFWA_{\varpi}(\lambda_1, \lambda_2, \dots, \lambda_n) = GIVPFWA_{\varpi}(\lambda_1^*, \lambda_2^*, \dots, \lambda_n^*) \quad (10)$$

where $\lambda_n^*(j = 1,2, \dots, n)$ is a permutation of $\lambda_j(j = 1,2, \dots, n)$

ii) Idempotency If $\lambda_j = \lambda$ for all j, then

$$GIVPFWA_{\varpi}(\lambda_1, \lambda_2, \dots, \lambda_n) = \lambda \quad (11)$$

iii) Boundedness: If $\lambda_{max} = max_j(\lambda_j)$ and $\lambda_{min} = min_j(\lambda_j)$, then

$$\lambda_{min} \leq GIVPFWA_{\varpi}(\lambda_1, \lambda_2, \dots, \lambda_n) \leq \lambda_{max} \quad (12)$$

iv) Monotonicity: If $\lambda_j \leq \lambda_j^*$, then

$$GIVPFWA_{\varpi}(\lambda_1, \lambda_2, \dots, \lambda_n) \leq GIVPFWA_{\varpi}(\lambda_1^*, \lambda_2^*, \dots, \lambda_n^*) \quad (13)$$

Proof: Proof is easy so it is omitted here.

Definition 8. The GIVPFOWA operator can be defined as:

$$\mathrm{GIVPFOWA}_{\varpi}\left(\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_n\right) = \left(\begin{array}{c} \left[\left(\sqrt{1-\prod_{j=1}^{n}\left(1-p_{\tilde{\lambda}_{\sigma(j)}}^{2\delta}\right)^{\varpi j}} \right)^{\frac{1}{\delta}}, \left(\sqrt{1-\prod_{j=1}^{n}\left(1-q_{\tilde{\lambda}_{\sigma(j)}}^{2\delta}\right)^{\varpi j}} \right)^{\frac{1}{\delta}} \right] \\ \left[\sqrt{1-\left[1-\prod_{j=1}^{n}\left(1-\left(1-r_{\tilde{\lambda}_{\sigma(j)}}^{2}\right)^{\delta}\right)^{\varpi j}\right]^{\frac{1}{\delta}}}, \sqrt{1-\left[1-\prod_{j=1}^{n}\left(1-\left(1-t_{\tilde{\lambda}_{\sigma(j)}}^{2}\right)^{\delta}\right)^{\varpi j}\right]^{\frac{1}{\delta}}} \right] \end{array} \right) \quad (14)$$

where $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$ be the weighted vector with conditions, $\varpi_J \in [0,1]$ and $\sum_{j=1}^{n} \varpi_j = 1$, , and $\delta > 0$.

Example5:

Let $\lambda_1 = ([0.3,0.6], [0.2,0.7]), \lambda_2 = ([0.4,0.5], [0.3,0.6]), \lambda_3 = ([0.2,0.6], [0.2,0.7])$, and $\lambda_4 = ([0.4,0.6], [0.4,0.5])$ and are four IVPFVs and let $\varpi = (0.4,0.2,0.1,0.3)^T$.First we find the score function, we have $S(\lambda_4) > S(\lambda_2) > S(\lambda_1) > S(\lambda_3)$. Now applying the GIVPFOWA operator, we get

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$$\mathrm{GIVPFOWA}_{\varpi}\left(\lambda_1,\lambda_2,\lambda_3,\lambda_4\right)=\left(\begin{array}{c}\left[\left(\sqrt{1-\prod_{j=1}^{4}\left(1-p_{\lambda_{\sigma(j)}}^{2\delta}\right)^{\varpi_j}}\right)^{\frac{1}{\delta}},\left(\sqrt{1-\prod_{j=1}^{4}\left(1-q_{\lambda_{\sigma(j)}}^{2\delta}\right)^{\varpi_j}}\right)^{\frac{1}{\delta}}\right]\\ \left[\sqrt{1-\left[1-\prod_{j=1}^{4}\left(1-\left(1-r_{\lambda_{\sigma(j)}}^{2}\right)^{\delta}\right)^{\varpi_j}\right]^{\frac{1}{\delta}}},\sqrt{1-\left[1-\prod_{j=1}^{4}\left(1-\left(1-t_{\lambda_{\sigma(j)}}^{2}\right)^{\delta}\right)^{\varpi_j}\right]^{\frac{1}{\delta}}}\right]\end{array}\right)$$

$$=\left[\left(0.343,0.603\right),\ \left(0.195,0.602\right)\right]$$

Definition 9: The GIVPFHA operator can be defined as:

$$\mathrm{GIVPFHA}_{\varpi,\varpi}\left(\lambda_1,\ldots,\lambda_n\right)=\left(\begin{array}{c}\left[\left(\sqrt{1-\prod_{j=1}^{n}\left(1-p_{\dot{\lambda}_{\sigma(j)}}^{2\delta}\right)^{\varpi_j}}\right)^{\frac{1}{\delta}},\left(\sqrt{1-\prod_{j=1}^{n}\left(1-q_{\dot{\lambda}_{\sigma(j)}}^{2\delta}\right)^{\varpi_j}}\right)^{\frac{1}{\delta}}\right]\\ \left[\sqrt{1-\left[1-\prod_{j=1}^{n}\left(1-\left(1-r_{\dot{\lambda}_{\sigma(j)}}^{2}\right)^{\delta}\right)^{\varpi_j}\right]^{\frac{1}{\delta}}},\sqrt{1-\left[1-\prod_{j=1}^{n}\left(1-\left(1-t_{\dot{\lambda}_{\sigma(j)}}^{2}\right)^{\delta}\right)^{\varpi_j}\right]^{\frac{1}{\delta}}}\right]\end{array}\right) \quad (15)$$

where $\lambda_{\sigma(j)}$ is the jth largest of the weighted interval-valued Pythagorean fuzzy values $\lambda_{\sigma(J)} = n\varpi_j\lambda_j$, $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$is the weighted vector with, $\varpi_J \in [0,1]$ and $\sum_{j=1}^{n}\varpi_j = 1$, and n is the balancing coefficient, which plays a role of balance. If the vector $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$approaches to $(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then we $(n\varpi_1\lambda_1, n\varpi_2\lambda_2, \dots, n\varpi_n\lambda_n)$ approach to$(\lambda_1, \lambda_2, \dots, \lambda_n)^T$.

Example6:

Let $\lambda_1 = ([0.2,0.4],[0.4,0.8]), \lambda_2 = ([0.4,0.6],[0.4,0.7]), \lambda_3 = ([0.1,0.4],[0.3,0.5])$ and $([0.2,0.6],[0.5,0.7])$ be the four IVPFVs whose weighted vector is $\varpi = (0.4,0.3,0.2,0.1)^T$and $\delta = 2$ then we have $\lambda_1 = ([0.224,0.449],[0.270,0.744]), \lambda_2 = ([0.418,0.625],[0.350,0.670]), \lambda_3 = ([0.094,0.378],[0.154,0.551])$ and $\lambda_4 = ([0.159,0.482],[0.685,0.643])$.Find the score functions, we have$S(\lambda_2) > S(\lambda_3) > S(\lambda_1) > S(\lambda_4)$. Thus we get the following result:

$$\mathrm{GIVPFHA}_{\varpi,\varpi}\left(\lambda_1,\lambda_2,\lambda_3,\lambda_4\right)=\left(\begin{array}{c}\left[\left(\sqrt{1-\prod_{j=1}^{4}\left(1-p^{2\delta}_{\dot{\lambda}_{\sigma(j)}}\right)^{\varpi j}}\right)^{\frac{1}{\delta}},\left(\sqrt{1-\prod_{j=1}^{4}\left(1-q^{2\delta}_{\dot{\lambda}_{\sigma(j)}}\right)^{\varpi j}}\right)^{\frac{1}{\delta}}\right]\\ \left[\sqrt{1-\left[1-\prod_{j=1}^{4}\left(1-\left(1-r^{2}_{\dot{\lambda}_{\sigma(j)}}\right)^{\delta}\right)^{\varpi j}\right]^{\frac{1}{\delta}}},\sqrt{1-\left[1-\prod_{j=1}^{4}\left(1-\left(1-t^{2}_{\dot{\lambda}_{\sigma(j)}}\right)^{\delta}\right)^{\varpi j}\right]^{\frac{1}{\delta}}}\right]\end{array}\right)$$

$$=\left([0.337,0.536],[0.273,0.639]\right)$$

An Application of the Proposed Aggregation Operators

Algorithm: Let $A=\{A_1,A_2,\ldots,A_n\}$ be a finite set of n options, and $C=(C_1,C_2,\ldots,C_m)$ be the set of m criteria and $\varpi=(\varpi_1,\varpi_2,\ldots,\varpi_m)^T$ be the weighted vector of the criteria $C_i(i=1,2,\ldots,m)$ such that $\varpi_i\epsilon[0,1]$ and $\sum_{i=1}^{m}\varpi_i=1$.

Step 1: The decision-maker provide has idea in the form of matrix.
Step 2: Compute $\alpha_j(j=1,2,\ldots,n)$ by applying the GIVPFWA aggregation operator.
Step 3: Calculate the score functions.
Step 4: Arrange the scores function of the all alternatives in the form of descending order and select that alternative, which has the highest score function value.

Illustrative Example

Suppose a customer wants to buy a car from different cars, let A_1,A_2,A_3,A_4 represent the four cars of different companies. Let C_1,C_2,C_3,C_4 be the criteria of these cars. C_1 : Price of each car, C_2 Model of each car, C_3 : Design of each car, C_4 : Color of each car. Suppose the weight vector of $C_i(i=1,2,3,4)$ is $=(0.4,0.3,0.2,0.1)^T$, and the interval-valued Pythagorean fuzzy values of the alternative $A_j(j=1,2,3,4)$ are denoted by the following decision matrix.

For GIVPFWA Aggregation Operator, for $\delta=2$

Step 1: The decision maker gives his decision in Table 1.

Table1.Pythagorean Fuzzy Decision Matrix

	A_1	A_2	A_3	A_4
C_1	([0.4,0.8], [0.4,0.6])	([0.4,0.6], [0.4,0.5])	([0.5,0.6], [0.2,0.7])	([0.2,0.4], [0.4,0.8])
C_2	([0.2,0.7], [0.2,0.5])	([0.3,0.6], [0.2,0.7])	([0.3,0.7], [0.4,0.7])	([0.4,0.6], [0.4,0.7])
C_3	([0.3,0.6], [0.3,0.7])	([0.3,0.5], [0.3,0.6])	([0.2,0.5], [0.2,0.6])	([0.1,0.4], [0.3,0.5])

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	A_1	A_2	A_3	A_4
C_4	([0.4,0.5], [0.4,0.8])	([0.2,0.6], [0.2,0.7])	([0.3,0.5], [0.5,0.8])	([0.2,0.6], [0.5,0.7])

Step 2: Applying GIVPFWA aggregation operator, we have

$$\alpha_1 = ([0.349,\ 0.726],\ [0.195,\ 0.602]),\ \alpha_2 = ([0.346,\ 0.584],\ [0.284,\ 0.589])$$
$$\alpha_3 = ([0.416,\ 0.617],\ [0.291,\ 0.719]),\ \alpha_4 = ([0.304,\ 0.511],\ [0.385,\ 0.679])$$

Step 3: Calculate the score functions of $\alpha_j (j = 1,2,3,4)$, we have
$$S(\alpha_1) \succ S(\alpha_2) \succ S(\alpha_3) \succ S(\alpha_4)$$

Step 4: Thus A_1 is the best option for customer.

For GIVPFOWA Aggregation Operator, for $\boldsymbol{\delta = 2}$

Step 1: Applying GIVPFOWA aggregation operator, we have

$$\alpha_1 = ([0.349,\ 0.726],\ [0.195,\ 0.602]), \alpha_2 = ([0.346,\ 0.584],\ [0.284,\ 0.589])$$
$$\alpha_3 = ([0.416,\ 0.617],\ [0.291,\ 0.719]), \alpha_4 = ([0.304,\ 0.511],\ [0.385,\ 0.679])$$

Step 2: Calculate the score functions of $\alpha_j\,(j = 1,2,3,4)$, we have
$$S(\alpha_1) \succ S(\alpha_2) \succ S(\alpha_3) \succ S(\alpha_4)$$

Step 3: Thus A_1 is the best option for customer.

For GIVPFHA Aggregation Operator, for $\boldsymbol{\delta = 2}$

Step 1: Applying $\dot{\alpha}_{\sigma(j)} = n\varpi_j \alpha_j$, we have

$$\dot{\alpha}_{11} = ([0.449,\ 0.827],\ [0.270,\ 0.495]), \dot{\alpha}_{21} = ([0.209,\ 0.727],\ [0.154,\ 0.454])$$
$$\dot{\alpha}_{31} = ([0.283,\ 0.569],\ [0.361,\ 0.643]), \dot{\alpha}_{41} = ([0.318,\ 0.399],\ [0.614,\ 0.876])$$
$$\dot{\alpha}_{12} = ([0.449,\ 0.668],\ [0.270,\ 0.378]), \alpha_{22} = ([0.313,\ 0.625],\ [0.154,\ 0.670])$$
$$\dot{\alpha}_{32} = ([0.283,\ 0.473],\ [0.361,\ 0.643]), \dot{\alpha}_{42} = ([0.159,\ 0.482],\ [0.127,\ 0.814])$$
$$\dot{\alpha}_{13} = ([0.559,\ 0.668],\ [0.092,\ 0.744]), \dot{\alpha}_{23} = ([0.313,\ 0.727],\ [0.350,\ 0.670])$$
$$\dot{\alpha}_{33} = ([0.189,\ 0.473],\ [0.259,\ 0.643]), \dot{\alpha}_{43} = ([0.238,\ 0.399],\ [0.685,\ 0.876])$$
$$\dot{\alpha}_{14} = ([0.224,\ 0.449],\ [0.270,\ 0.744]), \dot{\alpha}_{24} = ([0.418,\ 0.625],\ [0.350,\ 0.670])$$
$$\dot{\alpha}_{34} = ([0.094,\ 0.378],\ [0.154,\ 0.551]), \dot{\alpha}_{44} = ([0.159,\ 0.482],\ [0.685,\ 0.814])$$

Step 2: Applying GIVPFHA aggregation operator, we have

$$\alpha_1 = ([0.372,\ 0.747],\ [0.257,\ 0.530]), \alpha_2 = ([0.378,\ 0.614],\ [0.223,\ 0.526])$$
$$\alpha_3 = ([0.456,\ 0.652],\ [0.202,\ 0.707]), \alpha_4 = ([0.318,\ 0.520],\ [0.282,\ 0.665])$$

Step 3: Calculate the score functions of $\alpha_j (j = 1,2,3,4)$, we have $S(\alpha_1) \succ S(\alpha_2) \succ S(\alpha_3) \succ S(\alpha_4)$

Step 4: Thus A_1 is the best option for customer.

Table 2. Ranking of the alternative at different values of δ

δ	Rahman *et al.*, (2019)	Score functions	Proposed Methods	Score functions
2	GIVPFEWA GIVPFEOWA GIVPFEHA	$S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $A_1 > A_2 > A_3 > A_4$	GIVPFWA GIVPFOWA GIVPFHA	$S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $A_1 > A_2 > A_3 > A_4$
3	GIVPFEWA GIVPFEOWA GIVPFEHA	$S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $A_1 > A_2 > A_3 > A_4$	GIVPFWA GIVPFOWA GIVPFHA	$S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $A_1 > A_2 > A_3 > A_4$
5	GIVPFEWA GIVPFEOWA GIVPFEHA	$S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $A_1 > A_2 > A_3 > A_4$	GIVPFWA GIVPFOWA GIVPFHA	$S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_2) > S(\alpha_3) > S(\alpha_4)$ $A_1 > A_2 > A_3 > A_4$
10	GIVPFEWA GIVPFEOWA GIVPFEHA	$S(\alpha_1) > S(\alpha_3) > S(\alpha_2) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_3) > S(\alpha_2) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_3) > S(\alpha_2) > S(\alpha_4)$	GIVPFWA GIVPFOWA GIVPFHA	$S(\alpha_1) > S(\alpha_3) > S(\alpha_2) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_3) > S(\alpha_2) > S(\alpha_4)$ $S(\alpha_1) > S(\alpha_3) > S(\alpha_2) > S(\alpha_4)$

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$A_1 > A_3 > A_2 > A_4$	$A_1 > A_3 > A_2 > A_4$

6. Conclusion

In this paper we have to develop some generalized aggregation operators using IVPFNs. Firstly, we have developed three generalized aggregation operators namely, the GIVPFWA operator, the GIVPFOWA operator and the GIVPFHA operator**.** Finally, we have developed a method for multi-criteria group decision making based on the proposed operators, and the operational processes have illustrated in detail. The suggested methodology can be used for any type of selection problem involving any number of selection attributes. We ended the paper with an application of the new approach in a decision making problem. Finally, an illustrative example is given to show the decision steps of the proposed methods and to demonstrate their effectiveness.

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