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Application of the induced generalized averaging hybrid aggregation operators using interval-valued Pythagorean fuzzy environment

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Abstract Induced aggregation operators are more suitable for aggregating the individual preference relations into a collective fuzzy preference relation. Therefore the focus of our this paper is to develop some induced generalized aggregation operators using interval-valued Pythagorean fuzzy numbers, such as induced generalized interval-valued Pythagorean fuzzy ordered weighted averaging (I-GIVPFOWA) operator, induced generalized interval-valued Pythagorean fuzzy hybrid averaging (I-GIVPFHA) operator. Some desirable properties, such as idempotency, boundedness, and monotonicity corresponding to the proposed operators have been investigated. The main advantage of the proposed operators is that these operators are able to reflect the complex attitudinal character of the decision-maker using order inducing variables and provide much more complete information for decision-making. Furthermore, these operators are applied to decision-making problems in which experts provide their preferences in the Pythagorean fuzzy environment to show the validity, practicality, and effectiveness of the new approach.

Keywords I-GIVPFOWA operator, I-GIVPFHA operator, Group decision making

1. Introduction

Intuitionistic fuzzy set Atanassov, (1986) theory is one of the successful extensions of the fuzzy set theory Zadeh, (1965), which is characterized by the degree of membership and degree of non-membership has been presented. Later on, Atanassov and Gargov, (1989) extended it to the interval-valued intuitionistic fuzzy sets (IVIFS), which is characterized by a membership degree and a non-membership degree, whose values are intervals rather than real numbers. Over the last four decades, the IFS and IVIFS have received more and more attention by introducing the various kinds of operators, information measures and employed them to solve the decision-making problems under

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the different environment (Garg,(2016);Garg,(2016); Su et al.,(2011); Kumar and Garg,(2016); Wei and Wang,(2007); Xu and Jain ,(2007); Wang et al.,(2009)). However, apart from these, Xu,(2010); Tan and Chen,(2010); Garg et al.,(2017), used the Choquet integral to develop some intuitionistic fuzzy aggregation operators, which not only consider the importance of the elements or their ordered positions, but also can reflect the correlations among the elements or their ordered positions. But the limitation of their studies is that they are valid only for those environments whose degrees sum is less than one. However, in day-to-day life, there are many situations where this condition is ruled out. For instance, if a person gives their preference in the form of membership and non-membership degrees towards a particular object is 0.8 and 0.6, and then clearly this situation is not handling with IFS. In order to resolve it, (Yager,(2013); Yager,(2014)) proposed the Pythagorean fuzzy set by relaxing this sum condition to its square sum less than one. For instance, corresponding to the above-considered example, we see that $(0.8)^2 + (0.6)^2 = 1$ and hence PFS is an extension of the existing IFS. After their pioneer work, Yager and Abbasov ,(2013), studied the relationship between the Pythagorean numbers and the complex numbers. Yager and Abbasov ,(2013)introduced the notion of the PFWA operator and PFOWA operator. Zeng and Xu ,(2014) introduced the notion of TOPSIS method using Pythagorean fuzzy numbers. Peng and Yang, (2015) developed some important results for Pythagorean fuzzy sets. (Garg ,(2016); Garg ,(2017)) used the Einstein sum and Einstein product and introduced the notion of Pythagorean fuzzy Einstein arithmetic aggregation operators and Pythagorean fuzzy Einstein geometric aggregation operators such as, PFEWA operator, PFEOWA operator, GPFEWA operator, GPFEOWA operator, PFEWG operator, PFEOWG operator, GPFEWG operator and GPFEOWG operator and also applied them to group decision making. Garg,(2017), further, presented some aggregation operators based on confidence level.

However, in some real decision-making problems, due to insufficiency in available information, it may be difficult for decision makers to exactly quantify their opinions with a crisp number, but they can be represented by an interval number within [0, 1]. Therefore it is so important to present the idea of interval-valued Pythagorean fuzzy sets (IVPFSs), which permit the membership degrees and non- membership degrees to a given set to have an interval value. Zhang, (2016) introduced the concept of interval-valued Pythagorean fuzzy set. Peng and Yang ,(2015) introduced the notion of, interval-valued Pythagorean fuzzy weighted averaging (IVPFWA) operator, interval-valued Pythagorean fuzzy weighted geometric (IVPFWG) operator and also introduced some of their fundamental and important properties. Garg, (2016) presented an interval-valued Pythagorean fuzzy weighted average (IPFWA) and interval-valued Pythagorean fuzzy weighted geometric (IPFWG) operators for solving the decision-making problem under IVPFS environment. Also, a novel accuracy function has been defined in it for ranking the different interval-valued Pythagorean fuzzy numbers (IVPFNs). Now, in order to compare the interval numbers, some score, as well as accuracy function, have been taken for measurement and then applied to solve MCDM problems. Garg,(2016) defined the concepts of correlation and correlation coefficients of PFSs. Garg,(2017) also presented an improved accuracy functions under the IVPFS for solving the decision-making

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problems. Rahman et al. (Rahman *et al*., (2017); Rahman *et al*., (2018); Rahman *et al*., (2019); Rahman *et al*., (2019); Rahman *et al*., (2019); Rahman *et al*., (2019); Rahman *et al*., (2020)) introduced the concept of many operators using PFVs and IVPFVs, such as PFEWG operator, IVPFOWA operators, IVPFHA operator, IVPFOWG operators, IVPFHG operator, I-IVPFOWA operator, I-IVPFHA operator, IVPFEWG operator, IVPFEOWG operator, IVPFEWG operator, GIVPFWG operator, GIVPFOWG operator, GIVPFHG operator. (Wang *et al*.,(2021); Wang and Garg, (2021); Wang and Li ,(2020)) developed several aggregation operators and applied them on group decision making problems.

Motivated from (Rahman *et al*.,(2019)), in which the authors developed some induced aggregation operators such as, I-IVPFOWA operator and I-IVPFHA operator. But in this paper we develop I-GIVPFOWA operator and I-GIVPFHA operator. The new proposed operators are the generalization of the methods developed in (Rahman *et al*.,(2019)). The operators proposed in this paper are more general and more flexible. Therefore the proposed operators provide more accurate and precise results as compare to the existing methods. Of course, superficially, it is more complicated in calculation. However, in real applications, we need to assign the specific parameter δ, first.

The remainder paper can be constructed as. In Section 2, we present some basic definition and results which will be used in our later sections. In Section 3, we develop I-GIVPFOWA operator and I-GIVPFHA operator. In Section 4, we develop application of the proposed operators. In Section 5, we construct a numerical example. In Section 6, we have conclusion.

2. Preliminaries

Definition 1: (Peng andYang, (2015))Let K be a universal set, then IVPFSs I in K can be defined as:

$$I = \{\langle k, \mu_I(k), v_I(k)\rangle | k \in K\} \qquad (1)$$

Where

$$\mu_I(k) = \left[\mu_I^a(k), \mu_I^b(k)\right] \subset [0,1], v_I(k) = \left[v_I^a(k), v_I^b(k)\right] \subset [0,1]$$

,such that $\mu_I^a(k) = \inf\left(\mu_I(k)\right)$,

$\mu_I^b(k) = \sup\left(\mu_I(k)\right), v_I^a(k) = \inf\left(v_I(k)\right), v_I^b(k) = \sup\left(v_I(k)\right)$ and

$0 \le \left(\mu_I^b(k)\right)^2 + \left(v_I^b(k)\right)^2 \le 1$.

Also $\pi_I(k) = \left[\pi_I^a(k), \pi_I^b(k)\right]$, for all $k \in K$.

Definition2: (Peng andYang,(2015)) Let $\lambda = ([p_\lambda, q_\lambda], [r_\lambda, t_\lambda])$ be an IVPFV, then $S(\lambda) = \frac{1}{2}[(p_\lambda)^2 + (q_\lambda)^2 - (r_\lambda)^2 - (t_\lambda)^2]$

and $H(\lambda) = \frac{1}{2}[(p_\lambda)^2 + (q_\lambda)^2 + (r_\lambda)^2 + (t_\lambda)^2]$be the scores function and accuracy function of λ respectively. If λ_1, λ_2 be two IVPFVs, then we have the following:

Note:$S(\lambda_1) < S(\lambda_2)$

1) If $S(\lambda_1) < S(\lambda_2)$Then $(\lambda_1) < (\lambda_2$
2) If $S(\lambda_1) = S(\lambda_2)$ then
(i) If $H(\lambda_1) = H(\lambda_2)$ then $(\lambda_1) = (\lambda_2)$
(ii) If $H(\lambda_1) < H(\lambda_2)$ then $(\lambda_1) < (\lambda_2)$

Definition 3: (Rahman *et al*., (2019)) Let $\langle u_j, \lambda_j\rangle (j = 1,2)$ be a collection of 2-tuple, and $\delta > 0$, then the following operational laws always hold:

$$\lambda_1 \oplus \lambda_2 = \left(\left[\sqrt{(p_{\lambda_1})^2 + (p_{\lambda_2})^2 - (p_{\lambda_1})^2 (p_{\lambda_2})^2}, \sqrt{(q_{\lambda_1})^2 + (q_{\lambda_2})^2 - (q_{\lambda_1})^2 (q_{\lambda_2})^2}\right], [r_{\lambda_1} r_{\lambda_2}, t_{\lambda_1} t_{\lambda_2}]\right) \tag{2}$$

$$\lambda_1 \otimes \lambda_2 = ([p_{\lambda_1} p_{\lambda_2}, q_{\lambda_1} q_{\lambda_2}], \left[\sqrt{(r_{\lambda_1})^2 + (r_{\lambda_2})^2 - (r_{\lambda_1})^2 (r_{\lambda_2})^2}, \sqrt{(t_{\lambda_1})^2 + (t_{\lambda_2})^2 - (t_{\lambda_1})^2 (t_{\lambda_2})^2}\right]) \tag{3}$$

$$\delta\lambda_1 = \left(\left[\sqrt{1 - (1 - (p_{\lambda_1})^2)^\delta}, \sqrt{1 - (1 - (q_{\lambda_1})^2)^\delta}\right], [(r_{\lambda_1})^\delta, (t_{\lambda_1})^\delta]\right) \tag{4}$$

$$(\lambda_1)^\delta = \left(\left[\left(p_{\lambda_1}\right)^\delta, (q_{\lambda_1})^\delta\right], \left[\sqrt{1 - (1 - (r_{\lambda_1})^2)^\delta}, \sqrt{1 - (1 - (t_{\lambda_1})^2)^\delta}\right]\right) \tag{5}$$

Definition 4 : (Rahman *et al*., (2019)) the I-IVPFOWA operator can be defined as:

$$\begin{aligned} &I - IVPFOWA_\varpi(\langle u_1, \lambda_1\rangle, \langle u_2, \lambda_2\rangle, \ldots, \langle u_n, \lambda_n\rangle) \\ &= \left(\left[\sqrt{1 - \prod_{j=1}^{n}\left(1 - \left(p_{\lambda_{\sigma(j)}}\right)^2\right)^{\varpi_j}}, \sqrt{1 - \prod_{j=1}^{n}\left(1 - \left(q_{\lambda_{\sigma(j)}}\right)^2\right)^{\varpi_j}}\right], \left[\prod_{j=1}^{n} (r_{\lambda_{\sigma(j)}})^{\varpi_j}, \prod_{j=1}^{n} (t_{\lambda_{\sigma(j)}})^{\varpi_j}\right]\right) \end{aligned} \tag{6}$$

where $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_n)^T$ is the weighted vector of $\lambda_j (j = 1,2, \ldots, n)$, with some conditions such as, $\varpi_j \in [0,1]$, and $\sum_{j=1}^{n} \varpi_j = 1$. Also $\lambda_{\sigma(j)}$ is the λ_j value of the IVPFOWA pairs $\langle u_j, \lambda_j\rangle$having the jth largest u_j and u_j in $\langle u_j, \lambda_j\rangle$ is referred to as the order inducing variable and λ_j as the Pythagorean fuzzy argument variable.

Example 1:

If $\langle u_1, \lambda_1\rangle = \langle 0.5, ([0.4,0.6], [0.3,0.7])\rangle, \langle u_2, \lambda_2\rangle = \langle 0.3, ([0.3,0.6], [0.2,0.7])\rangle$, $\langle u_3, \lambda_3\rangle = \langle 0.6, ([0.3,0.8], [0.3,0.5])\rangle$,and $\langle u_4, \lambda_4\rangle = \langle 0.8, ([0.4,0.9], [0.1,0.3])\rangle$ be the four IVPFVs and let $\varpi = (0.1,0.2,0.3,0.4)^T$ be the weighted vector. Performing the ordering of the IPFOWA pairs with respect to the first component, we have $\langle u_4, \lambda_4\rangle = \langle 0.8, ([0.4,0.9], [0.1,0.3])\rangle, \langle u_3, \lambda_3\rangle = \langle 0.6, ([0.3,0.8], [0.3,0.5])\rangle, \langle u_1, \lambda_1\rangle = \langle 0.5, ([0.4,0.6], [0.3,0.7])\rangle, \langle u_2, \lambda_2\rangle = \langle 0.3, ([0.3,0.6], [0.2,0.7])\rangle$.

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This ordering includes the ordered interval-valued intuitionistic fuzzy arguments

$$\left\langle u_{\sigma(1)}, \tilde{\lambda}_{\sigma(1)} \right\rangle = \left\langle 0.8, \left([0.4, 0.9], [0.1, 0.3]\right) \right\rangle,$$

$$\left\langle u_{\sigma(2)}, \tilde{\lambda}_{\sigma(2)} \right\rangle = \left\langle 0.6, \left([0.3, 0.8], [0.3, 0.5]\right) \right\rangle,$$

$$\left\langle u_{\sigma(3)}, \tilde{\lambda}_{\sigma(3)} \right\rangle = \left\langle 0.5, \left([0.4, 0.6], [0.3, 0.7]\right) \right\rangle$$

and

$$\left\langle u_{\sigma(4)}, \tilde{\lambda}_{\sigma(4)} \right\rangle = \left\langle 0.3, \left([0.3, 0.6], [0.2, 0.7]\right) \right\rangle.$$

Now applying the I-IVPFOWA operator, we have

$$\text{I-IVPFOWA}_{\varpi}\left(\langle u_1, \tilde{\lambda}_1 \rangle, \langle u_2, \tilde{\lambda}_2 \rangle, \langle u_3, \tilde{\lambda}_3 \rangle, \langle u_4, \tilde{\lambda}_4 \rangle\right)$$
$$= \left(\left[\sqrt{1 - \prod_{j=1}^{4} \left(1 - p^2_{\tilde{\lambda}_{\sigma(j)}}\right)^{\varpi_j}}, \sqrt{1 - \prod_{j=1}^{4} \left(1 - q^2_{\tilde{\lambda}_{\sigma(j)}}\right)^{\varpi_j}} \right], \left[\prod_{j=1}^{4} \left(r_{\tilde{\lambda}_{\sigma(j)}}\right)^{\varpi_j}, \prod_{j=1}^{4} \left(t_{\tilde{\lambda}_{\sigma(j)}}\right)^{\varpi_j} \right] \right)$$
$$= \left([0.344, 0.703], [0.228, 0.601]\right).$$

Definition 5 : (Rahman *et al*., (2019))The I-IVPFHA operator can be defined as:

$$I - IVPFHA_{\varpi,\varpi}(\langle u_1, \lambda_1 \rangle, \langle u_2, \lambda_2 \rangle, \dots, \langle u_n, \lambda_n \rangle)$$
$$= \left(\left[\sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(p_{\lambda_{\sigma(j)}}\right)^2\right)^{\varpi_j}}, \sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(q_{\lambda_{\sigma(j)}}\right)^2\right)^{\varpi_j}} \right], \left[\prod_{j=1}^{n} (r_{\lambda_{\sigma(j)}})^{\varpi_j}, \prod_{j=1}^{n} (t_{\lambda_{\sigma(j)}})^{\varpi_j} \right] \right) \tag{7}$$

Where $\lambda_{\sigma(j)}$ is the jth largest of the weighted interval-valued Pythagorean fuzzy values $\lambda_{\sigma(j)} = n\varpi_j\lambda_j$, $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$ is the weighted vector of $\lambda_{\sigma(j)}$ and also $\varpi_J \in [0,1]$ and $\sum_{j=1}^{n} \varpi_j = 1$, and n is the balancing coefficient, which plays a role of balance. If $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$ approaches to $(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then $(n\varpi_1\lambda_1, n\varpi_2\lambda_2, \dots, n\varpi_n\lambda_n)$approaches to $(\lambda_1, \lambda_2, \dots, \lambda_n)^T$.

Example 2: If
$\langle u_1, \tilde{\lambda}_1 \rangle = \left\langle 0.3, \left([0.4, 0.7], [0.3, 0.4]\right) \right\rangle, \langle u_2, \tilde{\lambda}_2 \rangle = \left\langle 0.4, \left([0.3, 0.6], [0.2, 0.4]\right) \right\rangle$
$\langle u_3, \tilde{\lambda}_3 \rangle = \left\langle 0.5, \left([0.3, 0.7], [0.3, 0.5]\right) \right\rangle$ and $\langle u_4, \tilde{\lambda}_4 \rangle = \left\langle 0.9, \left([0.4, 0.8], [0.1, 0.3]\right) \right\rangle$ be the four IVPFVs whose weighted vector is $\varpi = (0.4, 0.3, 0.2, 0.1)^T$, then we have

$$\dot{\tilde{\lambda}}_1 = \left([0.259, 0.485], [0.617, 0.693]\right)\ \dot{\tilde{\lambda}}_2 = \left([0.269, 0.547], [0.275, 0.480]\right),$$

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$\dot{\lambda}_3 = ([0.327, 0.744], [0.235, 0.435])$, $\dot{\lambda}_4 = ([0.493, 0.897], [0.025, 0.145])$.

Performing the ordering with respect to the first element, we have

$\langle u_4, \lambda_4 \rangle = \langle 0.9, ([0.4, 0.8], [0.1, 0.3]) \rangle$, $\langle u_3, \lambda_3 \rangle = \langle 0.5, ([0.3, 0.7], [0.3, 0.5]) \rangle$,
$\langle u_2, \lambda_2 \rangle = \langle 0.4, ([0.3, 0.6], [0.2, 0.4]) \rangle$ and $\langle u_1, \lambda_1 \rangle = \langle 0.3, ([0.4, 0.7], [0.3, 0.4]) \rangle$.

Hence we get

$$\left\langle u_{\sigma(1)}, \dot{\lambda}_{\sigma(1)} \right\rangle = ([0.493, 0.897], [0.025, 0.145])$$

$$\left\langle u_{\sigma(2)}, \dot{\lambda}_{\sigma(2)} \right\rangle = ([0.327, 0.744], [0.235, 0.435]),$$

$$\left\langle u_{\sigma(3)}, \dot{\lambda}_{\sigma(3)} \right\rangle = ([0.269, 0.547], [0.275, 0.480])$$

and

$$\left\langle u_{\sigma(4)}, \dot{\lambda}_{\sigma(4)} \right\rangle = ([0.259, 0.485], [0.617, 0.693]).$$

Applying the I-IVPFHA operator and get

$$\text{I-IVPFHA}_{\varpi,\varpi}\left(\langle u_1, \lambda_1 \rangle, \langle u_2, \lambda_2 \rangle, \langle u_3, \lambda_3 \rangle, \langle u_4, \lambda_4 \rangle\right)$$
$$= \left(\left[\sqrt{1 - \prod_{j=1}^{4} \left(1 - p^2_{\dot{\lambda}_{\sigma(j)}} \right)^{\varpi_j}}, \sqrt{1 - \prod_{j=1}^{4} \left(1 - q^2_{\dot{\lambda}_{\sigma(j)}} \right)^{\varpi_j}} \right], \left[\prod_{j=1}^{4} \left(r_{\dot{\lambda}_{\sigma(j)}} \right)^{\varpi_j}, \prod_{j=1}^{4} \left(t_{\dot{\lambda}_{\sigma(j)}} \right)^{\varpi_j} \right] \right)$$
$$= ([0.705, 0.793], [0.109, 0.300])$$

3. Some induced generalized averaging aggregation operators

In this section, we introduce the notion of two induced generalized interval-valued aggregation operators such as, I-GIVPFOWA operator I-GIVPFHA operator. We also discuss some desirable properties of these propose operators such as, idempotency, boundedness, commutatively, monotonicity.

Definition 6: The I-GIVPFOWA operator can be defined as:

$$I - GIVPFOWA_{\varpi}(\langle u_1, \lambda_1\rangle, \langle u_2, \lambda_2\rangle, \dots, \langle u_n, \lambda_n\rangle) = \left(\frac{\left[(\sqrt{1 - \prod_{j=1}^{n}\left(1 - p^{2\delta}_{\lambda_{\sigma(j)}}\right)^{\varpi_j}})^{\frac{1}{\delta}}, (\sqrt{1 - \prod_{j=1}^{n}\left(1 - q^{2\delta}_{\lambda_{\sigma(j)}}\right)^{\varpi_j}})^{\frac{1}{\delta}} \right]}{\left[\sqrt{1 - \left[1 - \prod_{j=1}^{n}((1-(1-r^{2}_{\lambda_{\sigma(j)}})^{\delta})^{\varpi_j}\right]^{\frac{1}{\delta}}}, \sqrt{1 - \left[1 - \prod_{j=1}^{n}((1-(1-t^{2}_{\lambda_{\sigma(j)}})^{\delta})^{\varpi_j}\right]^{\frac{1}{\delta}}} \right]} \right) \quad (8)$$

where $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$ is the weighted vector of $\lambda_j (j = 1,2, \dots, n)$, with some conditions such as, $\varpi_i \epsilon [0,1]$ and $\sum_{i=1}^{m} \varpi_i = 1$ Also $\lambda_{\sigma_{(j)}}$ is the λ_j value of the IVPFOWA pairs $\langle u_j, \lambda_j\rangle$having the j[th] largest u_j and u_j in $\langle u_j, \lambda_j\rangle$ is referred to as the order inducing variable and λ_j as the Pythagorean fuzzy argument variable, and $\delta > 0$

Example 3: Let

$$\langle u_1, \lambda_1\rangle = \langle 0.6, ([0.3, 0.5], [0.4, 0.8])\rangle, \langle u_2, \lambda_2\rangle = \langle 0.3, ([0.3, 0.7], [0.2, 0.5])\rangle$$
$$\langle u_3, \lambda_3\rangle = \langle 0.5, ([0.2, 0.6], [0.3, 0.7])\rangle \text{ and } \langle u_4, \lambda_4\rangle = \langle 0, 2, ([0.4, 0.5], [0.4, 0.6])\rangle$$

be the four IVPFVs and let $\varpi = (0.1,0.2,0.3,0.4)^T$ be the weighted vector o and $\delta = 2$ Performing the ordering of the IVPFOWA pairs with respect to the first component, we have

$$\langle u_1, \lambda_1\rangle = \langle 0.6, ([0.3, 0.5], [0.4, 0.8])\rangle, \langle u_3, \lambda_3\rangle = \langle 0.5, ([0.2, 0.6], [0.3, 0.7])\rangle,$$
$$\langle u_2, \lambda_2\rangle = \langle 0.3, ([0.3, 0.7], [0.2, 0.5])\rangle \text{ and } \langle u_4, \lambda_4\rangle = \langle 0, 2, ([0.4, 0.5], [0.4, 0.6])\rangle.$$

This ordering includes the ordered interval-valued Pythagorean fuzzy arguments:

$$\langle u_1, \lambda_1\rangle = \langle 0.6, ([0.3, 0.5], [0.4, 0.8])\rangle \ \langle u_3, \lambda_3\rangle = \langle 0.5, ([0.2, 0.6], [0.3, 0.7])\rangle,$$
$$\langle u_2, \lambda_2\rangle = \langle 0.3, ([0.3, 0.7], [0.2, 0.5])\rangle, \langle u_4, \lambda_4\rangle = \langle 0, 2, ([0.4, 0.5], [0.4, 0.6])\rangle.$$

Now applying the given operator, and get

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$$\text{I-GIVPFOWA}_{\varpi}\left(\langle u_1,\lambda_1\rangle,\langle u_2,\lambda_2\rangle,\langle u_3,\lambda_3\rangle,\langle u_4,\lambda_4\rangle\right)$$

$$=\left(\begin{array}{c}\left[\left(\sqrt{1-\prod_{j=1}^{4}\left(1-p_{\lambda_{\sigma(j)}}^{2\delta}\right)^{\varpi j}}\right)^{\frac{1}{\delta}},\left(\sqrt{1-\prod_{j=1}^{4}\left(1-q_{\lambda_{\sigma(j)}}^{2\delta}\right)^{\varpi j}}\right)^{\frac{1}{\delta}}\right]\\ \left[\sqrt{1-\left[1-\prod_{j=1}^{4}\left(1-\left(1-r_{\lambda_{\sigma(j)}}^{2}\right)^{\delta}\right)^{\varpi j}\right]^{\frac{1}{\delta}}},\sqrt{1-\left[1-\prod_{j=1}^{4}\left(1-\left(1-t_{\lambda_{\sigma(j)}}^{2}\right)^{\delta}\right)^{\varpi j}\right]^{\frac{1}{\delta}}}\right]\end{array}\right)$$

$$=\left([0.34,0.60],[0.19,0.60]\right)$$

Theorem 1: Let $\langle u_j,\lambda_j\rangle (j=1,2,\dots,n)$ be a collection of 2-tuples, then their aggregated value by using the I-GIVPFOWA operator is also an IVPFV, and

$$I-GIVPFOWA_{\varpi}(\langle u_1,\lambda_1\rangle,\langle u_2,\lambda_2\rangle,\dots,\langle u_n,\lambda_n\rangle)$$
$$=\left(\frac{\left[(\sqrt{1-\prod_{j=1}^{n}\left(1-p_{\lambda_{\sigma(j)}}^{2\delta}\right)^{\varpi_j}})^{\frac{1}{\delta}},(\sqrt{1-\prod_{j=1}^{n}\left(1-q_{\lambda_{\sigma(j)}}^{2\delta}\right)^{\varpi_j}})^{\frac{1}{\delta}}\right]}{\left[\sqrt{1-\left[1-\prod_{j=1}^{n}(1-(1-r_{\lambda_{\sigma(j)}}^{2})^{\delta})^{\varpi_j}\right]^{\frac{1}{\delta}}},\sqrt{1-\left[1-\prod_{j=1}^{n}(1-(1-t_{\lambda_{\sigma(j)}}^{2})^{\delta})^{\varpi_j}\right]^{\frac{1}{\delta}}}\right]}\right) \tag{9}$$

Theorem 2: Let $\langle u_j,\lambda_j\rangle (j=1,2,\dots,n)$ and $\langle u_j{}^*,\lambda_j{}^*\rangle (j=1,2,\dots,n)$ be two set of 2-tuples, then

$$\mathrm{I-GIVPFOWA}_{\varpi}(\langle u_1,\lambda_1\rangle,\langle u_2,\lambda_2\rangle,\dots,\langle u_n,\lambda_n\rangle) = \mathrm{I-GIVPFOWA}_{\varpi}(\langle u_1,\lambda_1^*\rangle,\langle u_2,\lambda_2^*\rangle,\dots,\langle u_n,\lambda_n^*\rangle) \tag{10}$$

where $\langle u_j{}^*,\lambda_j{}^*\rangle (j=1,2,\dots,n)$ is a permutation of $\langle u_j,\lambda_j\rangle (j=1,2,\dots,n)$ and $\varpi=(\varpi_1,\varpi_2,\dots,\varpi_n)^T$ be the weighted vector with some conditions such as, $\varpi_i \epsilon [0,1]$, $\sum_{i=1}^{m}\varpi_i=1$.

Theorem 3: If $\langle u_j,\lambda_j\rangle (j=1,2,\dots,n)$ be a collection of 2-tuples, where $\lambda_{\sigma_{(j)}}=\lambda$ for all j, then

$$\mathrm{I-GIVPFOWA}_{\varpi}(\langle u_1,\lambda_1\rangle,\langle u_2,\lambda_2\rangle,\dots,\langle u_n,\lambda_n\rangle)=\lambda \tag{11}$$

Theorem 4: Let $\langle u_j,\lambda_j\rangle (j=1,2,\dots,n)$ be a collection of 2-tuples whose weighted vector is given by $\varpi=(\varpi_1,\varpi_2,\dots,\varpi_m)^T$, then

$$\lambda_{\min}\le \mathrm{I-GIVPFOWA}_{\varpi}(\langle u_1,\lambda_1\rangle,\langle u_2,\lambda_2\rangle,\dots,\langle u_n,\lambda_n\rangle)\le\lambda_{\max} \tag{12}$$

Where $\lambda_{max} = max_j(\lambda_j), \lambda_{min} = min_j(\lambda_j)$

Theorem 5: Let $\langle u_j, \lambda_j \rangle (j = 1,2, \dots, n)$ and $\langle u_j, \lambda_j{}^* \rangle (j = 1,2, \dots, n)$ where $\lambda_j \leq \lambda_j{}^*$ for all j, then

$$\mathrm{I-GIVPFOWA}_{\varpi}(\langle u_1, \lambda_1 \rangle, \langle u_2, \lambda_2 \rangle, \dots, \langle u_n, \lambda_n \rangle) \leq \mathrm{I-GIVPFOWA}_{\varpi}(\langle u_1, \lambda_1^* \rangle, \langle u_2, \lambda_2^* \rangle, \dots, \langle u_n, \lambda_n^* \rangle) \tag{13}$$

Definition 7: The I-GIVPFHA operator can be defined as:

$$I - GIVPFHA_{\varpi,\varpi}(\langle u_1, \lambda_1 \rangle, \langle u_2, \lambda_2 \rangle, \dots, \langle u_n, \lambda_n \rangle) = \left(\frac{\left[\left(\sqrt{1 - \prod_{j=1}^{n} \left(1 - p_{\dot{\lambda}_{\sigma(j)}}^{2\delta}\right)^{\varpi_j}}\right)^{\frac{1}{\delta}}, \left(\sqrt{1 - \prod_{j=1}^{n} \left(1 - q_{\dot{\lambda}_{\sigma(j)}}^{2\delta}\right)^{\varpi_j}}\right)^{\frac{1}{\delta}} \right]}{\left[\sqrt{1 - \left[1 - \prod_{j=1}^{n}(1 - (1 - r_{\dot{\lambda}_{\sigma(j)}}^2)^{\delta})^{\varpi_j}\right]^{\frac{1}{\delta}}}, \sqrt{1 - \left[1 - \prod_{j=1}^{n}(1 - (1 - t_{\dot{\lambda}_{\sigma(j)}}^2)^{\delta})^{\varpi_j}\right]^{\frac{1}{\delta}}} \right]} \right) \tag{14}$$

where $\lambda_{\sigma(j)}$ is the jth largest of the weighted interval-valued Pythagorean fuzzy values $\lambda_{\sigma(j)} \left(\lambda_{\sigma(j)} = n\varpi_j \lambda_j\right)$, $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$ is the weighted vector of $\lambda_{\sigma(j)}$ and also $\varpi_J \in [0,1]$, $\sum_{j=1}^{n} \varpi_j = 1$, and n is the balancing coefficient, which plays a role of balance. If the vector $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$ approaches to $(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then $(n\varpi_1\lambda_1, n\varpi_2\lambda_2, \dots, n\varpi_n\lambda_n)$ approaches to $(\lambda_1, \lambda_2, \dots, \lambda_n)^T$ and $\delta > 0$.

Example 4: Let

$\langle u_1, \lambda_1 \rangle = \langle 0.7, ([0.3, 0.6], [0.2, 0.7]) \rangle$, $\langle u_2, \lambda_2 \rangle = \langle 0.5, ([0.4, 0.5], [0.3, 0.6]) \rangle$
$\langle u_3, \lambda_3 \rangle = \langle 0.4, ([0.2, 0.6], [0.2, 0.7]) \rangle$ and $\langle u_4, \lambda_4 \rangle = \langle 0.1, ([0.4, 0.6], [0.4, 0.5]) \rangle$ be the four IVPFVs, let $\varpi = (0.1, 0.2, 0.3, 0.4)^T$ be the weighted vector and $\delta = 2$. Performing the ordering of the IVPFOWA pairs with respect to the first element, we have

$\dot{\lambda}_1 = ([0.238, 0.484], [0.127, 0.814])$ $\dot{\lambda}_2 = ([0.378, 0.573], [0.361, 0.643])$,

$\dot{\lambda}_3 = ([0.209, 0.624], [0.154, 0.670])$ and $\dot{\lambda}_4 = ([0.449, 0.669], [0.270, 0.378])$

Performing the ordering with respect to the first element, we have

$\langle u_1, \lambda_1 \rangle = \langle 0.7, ([0.3, 0.6], [0.2, 0.7]) \rangle$, $\langle u_2, \lambda_2 \rangle = \langle 0.5, ([0.4, 0.5], [0.3, 0.6]) \rangle$,
$\langle u_3, \lambda_3 \rangle = \langle 0.4, ([0.2, 0.6], [0.2, 0.7]) \rangle$ and $\langle u_4, \lambda_4 \rangle = \langle 0.1, ([0.4, 0.6], [0.4, 0.5]) \rangle$.

Hence we get

$$\left\langle u_{\sigma(1)}, \dot{\lambda}_{\sigma(1)} \right\rangle = \left([0.238, 0.484], [0.127, 0.814]\right),$$

$$\left\langle u_{\sigma(2)}, \dot{\lambda}_{\sigma(2)} \right\rangle = \left([0.378, 0.573], [0.361, 0.643]\right),$$

$$\left\langle u_{\sigma(3)}, \dot{\lambda}_{\sigma(3)} \right\rangle = \left([0.209, 0.624], [0.154, 0.670]\right)$$

And

$$\left\langle u_{\sigma(4)}, \dot{\lambda}_{\sigma(4)} \right\rangle = \left([0.449, 0.669], [0.270, 0.378]\right).$$

Thus

$$\text{I-GIVPFHA}_{\varpi,\varpi}\left(\langle u_1, \lambda_1\rangle, \langle u_2, \lambda_2\rangle, \langle u_3, \lambda_3\rangle, \langle u_4, \lambda_4\rangle\right)$$

$$= \left(\frac{\left[\left(\sqrt{1 - \prod_{j=1}^{4} \left(1 - p^{2\delta}_{\dot{\lambda}_{\sigma(j)}} \right)^{\varpi_j}} \right)^{\frac{1}{\delta}}, \left(\sqrt{1 - \prod_{j=1}^{4} \left(1 - q^{2\delta}_{\dot{\lambda}_{\sigma(j)}} \right)^{\varpi_j}} \right)^{\frac{1}{\delta}} \right]}{\left[\sqrt{1 - \left[1 - \prod_{j=1}^{4} \left(1 - \left(1 - r^{2}_{\dot{\lambda}_{\sigma(j)}} \right)^{\delta} \right)^{\varpi_j} \right]^{\frac{1}{\delta}}}, \sqrt{1 - \left[1 - \prod_{j=1}^{4} \left(1 - \left(1 - t^{2}_{\dot{\lambda}_{\sigma(j)}} \right)^{\delta} \right)^{\varpi_j} \right]^{\frac{1}{\delta}}} \right]} \right)$$

$$= \left([0.382, 0.614], [0.223, 0.526]\right)$$

Theorem 6: Let $\langle u_j, \lambda_j\rangle (j = 1,2, \dots, n)$ be a collection of 2-tuples, then their aggregated value by using the I-GIVPFHA operator is also an IVPFV, and

$$I - GIVPFHA_{\varpi,\varpi}(\langle u_1, \lambda_1\rangle, \langle u_2, \lambda_2\rangle, \dots, \langle u_n, \lambda_n\rangle)$$

$$= \left(\frac{\left[(\sqrt{1 - \prod_{j=1}^{n} \left(1 - p^{2\delta}_{\dot{\lambda}_{\sigma(j)}} \right)^{\varpi_j}})^{\frac{1}{\delta}}, (\sqrt{1 - \prod_{j=1}^{n} \left(1 - q^{2\delta}_{\dot{\lambda}_{\sigma(j)}} \right)^{\varpi_j}})^{\frac{1}{\delta}} \right]}{\left[\sqrt{1 - \left[1 - \prod_{j=1}^{n} (1 - (1 - r^2_{\dot{\lambda}_{\sigma(j)}})^{\delta})^{\varpi_j} \right]^{\frac{1}{\delta}}}, \sqrt{1 - \left[1 - \prod_{j=1}^{n} (1 - (1 - t^2_{\dot{\lambda}_{\sigma(j)}})^{\delta})^{\varpi_j} \right]^{\frac{1}{\delta}}} \right]} \right) \tag{15}$$

Theorem 7: If $\langle u_j, \lambda_j\rangle (j = 1,2, \dots, n)$ be a collection of 2-tuples, where $\lambda_{\sigma(j)} = \lambda$ for

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all j, then

$$I - GIVPFHA_{\varpi,\varpi}(\langle u_1, \lambda_1\rangle, \langle u_2, \lambda_2\rangle, \dots, \langle u_n, \lambda_n\rangle) = \lambda \quad (16)$$

Theorem 8: Let $\langle u_j, \lambda_j\rangle (j = 1,2, \dots, n)$ be a collection of 2-tuples whose weighted vector is given by $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$ then

$$\dot{\lambda}_{\min} \leq \mathrm{I} - \mathrm{GIVPFHA}_{\varpi,\varpi}(\langle \mathrm{u}_1, \lambda_1\rangle, \langle \mathrm{u}_2, \lambda_2\rangle, \dots, \langle \mathrm{u}_n, \lambda_n\rangle) \leq \dot{\lambda}_{\max} \quad (17)$$

Where $\lambda_{max} = max(\lambda_j), \lambda_{min} = min(\lambda_j)$.

Theorem 9: Let $\langle u_j, \lambda_j\rangle (j = 1,2, \dots, n)$ and $\langle u_j, {\lambda_j}^*\rangle (j = 1,2, \dots, n)$ where $\lambda_j \leq {\lambda_j}^*$ for all j, then

$$\begin{aligned} &I - GIVPFHA_{\varpi,\varpi}(\langle u_1, \lambda_1\rangle, \langle u_2, \lambda_2\rangle, \dots, \langle u_n, \lambda_n\rangle) \\ &\qquad \leq I - GIVPFHA_{\varpi,\varpi}(\langle u_1, \lambda_1^*\rangle, \langle u_2, \lambda_2^*\rangle, \dots, \langle u_n, \lambda_n^*\rangle) \end{aligned} \quad (18)$$

4. An application of the proposed aggregation operators

Let $A = \{A_1, A_2, \dots, A_n\}$ be a finite set of n options, and $C = (C_1, C_2, \dots, C_m)$ be a finite set of m criteria and $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_m)^T$ be the weighted vector of the criteria $C_i (i = 1,2, \dots, m)$ such that $\varpi_i \epsilon [0,1]$ and $\sum_{i=1}^{m} \varpi_i = 1$. The method proposed in this paper having the following steps.

Step 1: Decision-maker provides their idea in the form of matrix.
Step 2: Compute $\alpha_j (j = 1,2, \dots, n)$ by applying the proposed operator.
Step 3: Calculate the score functions.
Step 4: Arrange the scores function of the all alternatives in the form of descending order and select that alternative, which has the highest score function value.

5. Illustrative example

Suppose a customer wants to buy a Laptop from different Laptops, let A_1, A_2, A_3, A_4 represent the four Laptops of different companies. Let C_1, C_2, C_3, C_4 be the criteria of these Laptops. C_1Price of each Laptop, C_2 Model of each Laptop, C_3Design of each Laptops, C_4 Hard disc of each Laptops. Suppose the weight vector of $C_i (i = 1,2,3,4)$ is $\varpi = (0.1, 0.2, 0.3, 0.4)^T$, and the interval-valued Pythagorean fuzzy values of the alternative $A_j (j = 1,2,3,4)$ are denoted by the following decision matrix.

Step 1: The decision maker gives his decision in Table 1.

Table1.Pythagorean Fuzzy Decision Matrix

	A_1	A_2	A_3	A_4
C_1	$\langle 0.6, ([0.3,0.5], [0.4,0.5])\rangle$	$\langle 0.7, ([0.3,0.6], [0.2,0.7])\rangle$	$\langle 0.8, ([0.3,0.5], [0.5,0.8])\rangle$	$\langle 0.9, ([0.2,0.6], [0.5,0.7])\rangle$
C_2	$\langle 0.5, ([0.2,0.6], [0.3,0.7])\rangle$	$\langle 0.5, ([0.4,0.5], [0.3,0.6])\rangle$	$\langle 0.6, ([0.2,0.5], [0.2,0.6])\rangle$	$\langle 0.7, ([0.5,0.8], [0.3,0.5])\rangle$
C_3	$\langle 0.3, ([0.3,0.7], [0.2,0.5])\rangle$	$\langle 0.4, ([0.2,0.6], [0.2,0.7])\rangle$	$\langle 0.4, ([0.3,0.7], [0.4,0.7])\rangle$	$\langle 0.5, ([0.3,0.6], [0.4,0.7])\rangle$
C_4	$\langle 0.2, ([0.4,0.5], [0.4,0.6])\rangle$	$\langle 0.1, ([0.4,0.6], [0.4,0.5])\rangle$	$\langle 0.2, ([0.4,0.4], [0.2,0.8])\rangle$	$\langle 0.4, ([0.2,0.4], [0.4,0.8])\rangle$

Step 2: Applying the I-GIVPFOWA operator, we have the following

$$\alpha_1 = ([0.343,\ 0.603],\ [0.195,\ 0.602]), \alpha_2 = ([0.466,\ 0.584],\ [0.284,\ 0.589])$$
$$\alpha_3 = ([0.343,\ 0.570],\ [0.291,\ 0.719]), \alpha_4 = ([0.355,\ 0.629],\ [0.385,\ 0.679])$$

Step 3: Calculate the scores functions of $\alpha_j (j = 1,2, \dots, n)$, we have

$$S(\alpha_1) = \frac{1}{2}\left[(0.343)^2 + (0.603)^2 - (0.195)^2 - (0.602)^2\right] = 0.040$$

$$S(\alpha_2) = \frac{1}{2}\left[(0.466)^2 + (0.584)^2 - (0.284)^2 - (0.589)^2\right] = 0.063$$

$$S(\alpha_3) = \frac{1}{2}\left[(0.343)^2 + (0.570)^2 - (0.291)^2 - (0.719)^2\right] = -0.079$$

$$S(\alpha_4) = \frac{1}{2}\left[(0.355)^2 + (0.629)^2 - (0.385)^2 - (0.679)^2\right] = -0.043$$

Hence $S(\alpha_2) \succ S(\alpha_1) \succ S(\alpha_4) \succ S(\alpha_3)$

Step 4: Thus A_2 is the best option for customer.

For I-GIVPFHA Aggregation Operator, where $\boldsymbol{\delta = 2}$

Step 1: Applying $\dot{\alpha}_{\sigma(j)} = n\varpi_j \alpha_j$, we have

$$\dot{\alpha}_{11} = ([0.238,\ 0.569],\ [0.614,\ 0.876]), \dot{\alpha}_{21} = ([0.189,\ 0.569],\ [0.361,\ 0.643])$$
$$\dot{\alpha}_{31} = ([0.313,\ 0.727],\ [0.154,\ 0.454]), \dot{\alpha}_{41} = ([0.449,\ 0.559],\ [0.270,\ 0.495])$$
$$\dot{\alpha}_{12} = ([0.238,\ 0.482],\ [0.127,\ 0.814]), \dot{\alpha}_{22} = ([0.378,\ 0.473],\ [0.361,\ 0.643])$$
$$\dot{\alpha}_{32} = ([0.209,\ 0.625],\ [0.154,\ 0.670]), \dot{\alpha}_{42} = ([0.449,\ 0.669],\ [0.270,\ 0.378])$$
$$\dot{\alpha}_{13} = ([0.238,\ 0.399],\ [0.685,\ 0.876]), \dot{\alpha}_{23} = ([0.189,\ 0.473],\ [0.259,\ 0.643])$$
$$\dot{\alpha}_{33} = ([0.313,\ 0.727],\ [0.350,\ 0.670]), \dot{\alpha}_{43} = ([0.449,\ 0.449],\ [0.092,\ 0.744])$$
$$\dot{\alpha}_{14} = ([0.159,\ 0.482],\ [0.685,\ 0.643]), \dot{\alpha}_{24} = ([0.473,\ 0.765],\ [0.154,\ 0.551])$$
$$\dot{\alpha}_{34} = ([0.313,\ 0.625],\ [0.350,\ 0.670]), \dot{\alpha}_{44} = ([0.224,\ 0.449],\ [0.270,\ 0.744])$$

Step 2: Applying I-GIVPFHA aggregation operator, we have

$$\alpha_1 = ([0.375,\ 0.623],\ [0.257,\ 0.530]), \alpha_2 = ([0.382,\ 0.614],\ [0.223,\ 0.526])$$
$$\alpha_3 = ([0.375,\ 0.588],\ [0.202,\ 0.707]), \alpha_4 = ([0.344,\ 0.616],\ [0.282,\ 0.665])$$

Step 3: Calculate the scores functions of $\alpha_j (j = 1,2,3,4)$, we have

$$S(\alpha_1) = \frac{1}{2}\left[(0.375)^2 + (0.623)^2 - (0.257)^2 - (0.530)^2\right] = 0.090$$
$$S(\alpha_2) = \frac{1}{2}\left[(0.382)^2 + (0.614)^2 - (0.223)^2 - (0.526)^2\right] = 0.098$$
$$S(\alpha_3) = \frac{1}{2}\left[(0.375)^2 + (0.588)^2 - (0.202)^2 - (0.707)^2\right] = -0.027$$
$$S(\alpha_4) = \frac{1}{2}\left[(0.344)^2 + (0.616)^2 - (0.282)^2 - (0.665)^2\right] = -0.011$$

Hence $S(\alpha_2) \succ S(\alpha_1) \succ S(\alpha_4) \succ S(\alpha_3)$

Step 4: Thus A_2 is the best option for customer.

Table 2. Ranking of the alternative at different values of δ

δ	Rahman *et al.*, (2019)	Score functions	Proposed Methods	Score functions
1	I-IVPFOWA I-IVPFHA	$S(\alpha_2) > S(\alpha_1) > S(\alpha_4) > S(\alpha_3)$ $S(\alpha_2) > S(\alpha_1) > S(\alpha_4) > S(\alpha_3)$ $A_2 > A_1 > A_4 > A_3$	I-GIVPFOWA I-GIVPFHA	$S(\alpha_2) > S(\alpha_1) > S(\alpha_4) > S(\alpha_3)$ $S(\alpha_2) > S(\alpha_1) > S(\alpha_4) > S(\alpha_3)$ $A_1 > A_2 > A_3 > A_4$
2	I-IVPFOWA I-IVPFHA	$S(\alpha_2) > S(\alpha_1) > S(\alpha_4) > S(\alpha_3)$ $S(\alpha_2) > S(\alpha_1) > S(\alpha_4) > S(\alpha_3)$ $A_2 > A_1 > A_4 > A_3$	I-GIVPFOWA I-GIVPFHA	$S(\alpha_2) > S(\alpha_1) > S(\alpha_4) > S(\alpha_3)$ $S(\alpha_2) > S(\alpha_1) > S(\alpha_4) > S(\alpha_3)$ $A_2 > A_1 > A_4 > A_3$
3	I-IVPFOWA I-IVPFHA	$S(\alpha_2) > S(\alpha_1) > S(\alpha_4) > S(\alpha_3)$ $S(\alpha_2) > S(\alpha_1) > S(\alpha_4) > S(\alpha_3)$ $A_2 > A_1 > A_4 > A_3$	I-GIVPFOWA I-GIVPFHA	$S(\alpha_2) > S(\alpha_1) > S(\alpha_4) > S(\alpha_3)$ $S(\alpha_2) > S(\alpha_1) > S(\alpha_4) > S(\alpha_3)$ $A_2 > A_1 > A_4 > A_3$
5	I-IVPFOWA I-IVPFHA	$S(\alpha_2) > S(\alpha_1) > S(\alpha_4) > S(\alpha_3)$ $S(\alpha_2) > S(\alpha_1) > S(\alpha_4) > S(\alpha_3)$ $A_2 > A_1 > A_4 > A_3$	I-GIVPFOWA I-GIVPFHA	$S(\alpha_2) > S(\alpha_1) > S(\alpha_3) > S(\alpha_4)$ $S(\alpha_2) > S(\alpha_1) > S(\alpha_3) > S(\alpha_4)$ $A_2 > A_1 > A_3 > A_4$

6. Conclusions

The objective of this paper is to present some induced aggregation operators based on interval-valued Pythagorean fuzzy numbers and applied them to the multi-attribute group decision making problems where attribute values are the interval-valued Pythagorean fuzzy numbers. Firstly, we have developed two induced aggregation operators a long with their properties namely, induced generalized interval-valued Pythagorean fuzzy ordered weighted averaging (I-GIVPFOWA) aggregation operator and induced generalized interval-valued Pythagorean fuzzy hybrid averaging (I-GIVPFHA) aggregation operator respectively. Finally, we have developed a method for multi-criteria group decision making based on these operators, and the operational processes have illustrated in detail. The suggested methodology can be used for any type of selection problem involving any number of selection attributes. We ended the paper with an application of the new approach in a decision making problem.

In further research, it is necessary to give the applications of these operators to the other domains such as, Confidence levels, Hamacher operators, Power operators, Symmetric operator, Logarithmic operators, Dombi operators, Linguistic terms.

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