Annals of Optimization Theory and Practice

Volume 1, Number 1, 15-49

December 2020

DOI: 10.22121/aotp.2020.241689.1036

Some induced generalized Einstein aggregating operators and their application to group decision-making problem using intuitionistic fuzzy numbers

Rahman Khaista[1]✉ • Ayub Sanam[2] • Abdullah Saleem[3] • Yaqub Khan Muhammad[4]

[1] Department of Mathematics, Hazara University, Mansehra, Pakistan

[2] Department of Mathematics and Statistics, Riphah International University, Islamabad, Pakistan

[3] Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan

[4] Department of Mathematics and Statistics, Riphah International University, Islamabad, Pakistan

✉ khaista355@yahoo.com

(Received: July 29, 2020 / Accepted: September 2, 2020)

Abstract The paper aims to develop an idea of some inducing operators, namely induced intuitionistic fuzzy Einstein hybrid averaging operator, induced intuitionistic fuzzy Einstein hybrid geometric operator, induced generalized intuitionistic fuzzy Einstein hybrid averaging operator and induced generalized intuitionistic fuzzy Einstein hybrid geometric operator along with their wanted structure properties such as, monotonicity, idempotency and boundedness. The proposed operators are competent and able to reflect the complex attitudinal character of the decision maker by using order inducing variables and deliver more information to experts for decision-making. To show the legitimacy, practicality and effectiveness of the new operators, the proposed operators have been applied to decision making problems.

Keywords I-IFEHA operator; I-IFEHG operator; I-GIFEHA operator; I-GIFEHG operator; MAGDM problem

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

1. Introduction

Multi-attribute decision-making play an importance acts and role in much area our daily such as management, medical, engineering, business and economics. Commonly, it has been recognized and assumed that the data and information which touch the alternatives in form of criteria and weight are articulated in real numbers. However due to the complication of the system time-by-time, it is not easy to an expert to make a correct decision, as most of the model value during the process of decision-making filled with uncertainty. Thus in decision-making process, the decision and verdict information and data is mostly and often incomplete, indeterminate and inconsistent information. Therefore, Zadeh (1965) firstly developed and familiarize the concept fuzzy set (FS) which is more powerful and controlling to process fuzzy information in our daily life problems. However, there are some problems in this approach and tactic, because it has only element and member called membership function. To control and cover the above limitations, Atanassov (1986) and Atanassov (1989) familiarized the idea of intuitionistic fuzzy set (IFS) by adding and including a new element called non-membership function. Intuitionistic fuzzy set is the more suitable method and ways for the controlling and managing of those problems and difficulties which occur in the form of fuzzy set. After the development of intuitionistic fuzzy set several researchers, such as Garg (2020) introduced the notion of some averaging and geometric operators. Garg and Arora (2019) generalized intuitionistic fuzzy soft power aggregation operators. Garg and Kumar (2020) presented the possibility measure of interval-valued intuitionistic fuzzy set. Xu (2007) developed models for MAGDM problem. Yu *et al.* (2012) introduced the idea of prioritized operators for IVIFNs. Bustince and Burillo (1995) introduced the idea of interval-valued intuitionistic fuzzy numbers. Xu and Xia (2011) introduced the idea of induced aggregation operators. Tan (2011) introduced the notion of some generalized intuitionistic fuzzy geometric operators. Xia and Xu (2010) introduced the notion of similarity measures for intuitionistic fuzzy values. Li (2011) introduced the notion of some generalized ordered weighted averaging operators. Wei (2008) and Wei (2010) and Wei (2011) developed the maximizing deviation method, GRA method and gray relational analysis method for IFSs. Wei and Zhao (2011) explore the notion of Minimum deviation models. Wei *et al.* (2011) developed the notion of correlation coefficient to IVIFNs. Grzegorzewski (2004) find distances between intuitionistic fuzzy sets. Yager (2009) developed some aspects of intuitionistic fuzzy sets. Ye (2010) and Ye (2011) introduced the concept of weighted correlation coefficient and fuzzy cross entropy for IVIFSs. Garg (2016) and Garg (2019) introduced the idea of generalized intuitionistic fuzzy interactive geometric interaction operators and Intuitionistic fuzzy Hamacher aggregation operators with entropy weight and their applications. Garg and Kumar (2018) and Garg and Kumar (2019) introduced the idea of power geometric aggregation operators and a novel correlation coefficient of intuitionistic fuzzy sets. Garg and Rani (2019) introduced the idea of some generalized geometric aggregation operators for complex intuitionistic fuzzy sets. Xu (2007) and Xu and Yager (2006) presented the idea of some geometric and arithmetic operators using algebraic rules, namely intuitionstic

fuzzy hybrid averaging operator, intuitionstic fuzzy weighted averaging operator, intuitionstic fuzzy ordered weighted averaging operator, intuitionstic fuzzy hybrid geometric operator, intuitionstic fuzzy ordered weighted geometric operator, intuitionstic fuzzy weighted geometric operator, respectively along with their wanted properties and also applied these on multi-attributes group decision making problems in daily life to select the best alternative from all under consideration alternatives. Einstein product and sum are the batter ways from the algebraic laws. Xu (2011) familiarized the notion of intuitionstic fuzzy power operators and their application. Wang and Liu (2011) and Wang and Liu (2012) presented the concept of Einstein operators namely, intuitionistic fuzzy Einstein weighted geometric operator, intuitionistic fuzzy Einstein ordered weighted geometric operator, intuitionistic fuzzy Einstein weighted averaging operator, intuitionistic fuzzy Einstein ordered weighted averaging operator and utilized these operators to MAGDM problems. After the developing of Einstein operators, Zhao and Wei (2013) generalized them and presented the idea of intuitionistic fuzzy Einstein hybrid averaging operator, intuitionistic fuzzy Einstein hybrid geometric operator. Zhao *et al.* (2010) familiarized the idea of generalized intuitionistic fuzzy weighted averaging operator, generalized intuitionistic fuzzy ordered weighted averaging operator, generalized intuitionistic fuzzy hybrid averaging operator, generalized interval-valued intuitionistic fuzzy weighted averaging operator, generalized interval-valued intuitionistic fuzzy ordered weighted averaging operator, generalized interval-valued intuitionistic fuzzy hybrid averaging operator. Wei (2010) and Yu and Xu (2013) and Su *et al.* (2011) and Xu *et al.* (2013) introduced the notion of some new type's methods and utilized them to group decision making. Rahman *et al.* (2018) and Rahman *et al.* (2020) introduced some generalized intuitionistic fuzzy Einstein hybrid aggregation Operators and confidence levels and their application to group decision making problems. Jamil *et al.* (2020) introduced the notion of induced generalized interval-valued intuitionistic fuzzy Einstein operator and applied them on group decision making problem.

Motivated from (37), in which the authors used some algebraic operational laws and develop some algebraic operators for intuitionistic fuzzy values such as, induced intuitionistic fuzzy hybrid averaging operator, induced intuitionistic fuzzy hybrid geometric operator. But Einstein product and sum are the best ways from the algebraic operational laws. Therefore, in this paper we utilize the Einstein sum and product and propose some new methods such as, induced intuitionistic fuzzy Einstein hybrid averaging operator, induced intuitionistic fuzzy Einstein hybrid geometric operator, induced generalized intuitionistic fuzzy Einstein hybrid averaging operator, induced generalized intuitionistic fuzzy Einstein hybrid geometric operator. The new operators are competent to reflect the complex attitudinal character of the decision maker with the help of inducing variables and provide more information to decision maker. The proposed methods provide more general, more accurate and precise results are comparing to their existing methods. Therefore, these methods play a vital role in real world problems.

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

The rest of the paper is structured as follows. Section two, contains some basic results. Section three, contains new methods, such as induced intuitionistic fuzzy Einstein hybrid averaging operator, induced intuitionistic fuzzy Einstein hybrid geometric operator, induced generalized intuitionistic fuzzy Einstein hybrid averaging operator and induced generalized intuitionistic fuzzy Einstein hybrid geometric operator. Section four, contains multi-attributes group decision making problem. Section five, contains numerical example. Section six, contains conclusion.

2. Preliminaries

In this section, we present fuzzy set, intuitionistic fuzzy set, score function, accuracy function, some Einstein operational laws, induced intuitionistic fuzzy hybrid averaging operator and induced intuitionistic fuzzy hybrid geometric operator.

Definition 1: [1] Let Z be a general set, then FS can be defined as:

$$F = \left\{ \left\langle z, \xi_F(z) \right\rangle \mid z \in Z \right\} \tag{1}$$

where $\xi_F(z): Z \to [0,1]$ called membership function.

Definition 2: [2, 3] Let Z be a general set, then intuitionistic fuzzy set can be defined as:

$$I = \left\{ \left\langle z, \xi_I(z), \kappa_I(z) \right\rangle \mid z \in Z \right\} \tag{2}$$

where $\xi_F(z): Z \to [0,1]$, $\kappa_F(z): Z \to [0,1]$ with $0 \le \xi_I(z) + \kappa_I(z) \le 1, \forall z \in Z$. Let $\pi_I(z) = 1 - \xi_I(z) - \kappa_I(z)$ then $\pi_I(z)$ is called the degree of indeterminacy.

Definition 3: [31] Let $\Re = (\xi_\Re, \kappa_\Re)$ be an intuitionistic fuzzy value, then the score function and accuracy function of $\mathcal{R}$ can be defined as: $S(\Re) = \xi_\Re - \kappa_\Re$ and $H(\Re) = \xi_\Re + \kappa_\Re$ respectively.

Definition 4: [31] Let $\Re_1 = \left(\xi_{\Re_1}, \kappa_{\Re_1}\right)$ and $\Re_2 = \left(\xi_{\Re_2}, \kappa_{\Re_2}\right)$ be the two IFVs, then

1. If $S(\Re_2) > S(\Re_1)$, then $\Re_1 \prec \Re_2$
2. If $S(\Re_2) = S(\Re_1)$, then
1. If $H(\Re_2) = H(\Re_1)$, then, $\Re_1 = \Re_2$

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

2. If $H(\Re_2) < H(\Re_1)$, then $\Re_1 \succ \Re_2$

Definition 5: [31] Let $\langle u_j, \Re_j \rangle (j = 1,2)$ be a family of 2-tuples and $\sigma > 0$, then

$$\Re_1 \oplus_{\varepsilon} \Re_2 = \left\langle \frac{\xi_{\Re_1} + \xi_{\Re_2}}{1 + \xi_{\Re_1}\xi_{\Re_2}}, \frac{\kappa_{\Re_1}\kappa_{\Re_2}}{1 + \left(1 - \kappa_{\Re_1}\right)\left(1 - \kappa_{\Re_2}\right)} \right\rangle \tag{3}$$

$$\Re_1 \otimes_{\varepsilon} \Re_2 = \left\langle \frac{\xi_{\Re_1}\xi_{\Re_2}}{1 + \left(1 - \xi_{\Re_1}\right)\left(1 - \xi_{\Re_2}\right)}, \frac{\kappa_{\Re_1} + \kappa_{\Re_2}}{1 + \kappa_{\Re_1}\kappa_{\Re_2}} \right\rangle \tag{4}$$

$$(\Re)^{\sigma} = \left\langle \frac{2\left(\xi_{\Re}\right)^{\sigma}}{\left(2 - \xi_{\Re}\right)^{\sigma} + \left(\xi_{\Re}\right)^{\sigma}}, \frac{\left(1 + \kappa_{\Re}\right)^{\sigma} - \left(1 - \kappa_{\Re}\right)^{\sigma}}{\left(1 + \kappa_{\Re}\right)^{\sigma} + \left(1 - \kappa_{\Re}\right)^{\sigma}} \right\rangle \tag{5}$$

$$\sigma(\Re) = \left\langle \frac{\left(1 + \xi_{\Re}\right)^{\sigma} - \left(1 - \xi_{\Re}\right)^{\sigma}}{\left(1 + \xi_{\Re}\right)^{\sigma} + \left(1 - \xi_{\Re}\right)^{\sigma}}, \frac{2\left(\kappa_{\Re}\right)^{\sigma}}{\left(2 - \kappa_{\Re}\right)^{\sigma} + \left(\kappa_{\Re}\right)^{\sigma}} \right\rangle \tag{6}$$

Definition6: [37] The induced intuitionistic fuzzy hybrid averaging operator can be defined as:

$$\text{I-IFHA}_{\partial,\Im}\left(\langle u_1, \Re_1 \rangle, \langle u_2, \Re_2 \rangle, \ldots, \langle u_n, \Re_n \rangle\right) = \left\langle 1 - \prod_{j=1}^{n}\left(1 - \xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}, \prod_{j=1}^{n}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j} \right\rangle \tag{7}$$

Where $\dot{\Re}_{\delta(j)}$ be the weighted intuitionistic fuzzy value $\dot{\Re}_j \left(\dot{\Re}_j = n\partial_j \Re_j, j = 1,\ldots,n\right)$ of the IFOWA pair $\langle u_j, \Re_j \rangle$ having the jth largest u_j in $\langle u_j, \Re_j \rangle$. $\Im = \left(\Im_1, \ldots, \Im_n\right)^T$ be the associated vector of induced intuitionistic fuzzy hybrid averaging operator with $\Im_j \in [0,1]$ and $\sum_{j=1}^{n} \Im_j = 1$. $\partial = \left(\partial_1, \ldots, \partial_n\right)^T$ be the weighted vector with $\partial_j \in [0,1]$ and $\sum_{j=1}^{n} \partial_j = 1$.

Definition 7: [37] The induced intuitionistic fuzzy hybrid geometric operator can be defined as:

$$\text{I-IFHG}_{\partial,\Im}\left(\langle u_1, \Re_1 \rangle, \langle u_2, \Re_2 \rangle, \ldots, \langle u_n, \Re_n \rangle\right) = \left\langle \prod_{j=1}^{n}\left(\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}, 1 - \prod_{j=1}^{n}\left(1 - \kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j} \right\rangle \tag{8}$$

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

where $\dot{\Re}_{\delta(j)}$ is the weighted intuitionistic fuzzy value $\dot{\Re}_j\left(\dot{\Re}_j=\left(\Re_j\right)^{n\partial_j}, j=1,2,...,n\right)$ of the intuitionistic fuzzy ordered weighted geometric pair $\left\langle u_j,\Re_j\right\rangle$ having the jth largest u_j in $\left\langle u_j,\Re_j\right\rangle$. $\Im=\left(\Im_1,...,\Im_n\right)^T$ be the associated vector of I-IFHG operator with $\Im_j\in[0,1]$ and $\sum_{j=1}^{n}\Im_j=1$. $\partial=\left(\partial_1,...,\partial_n\right)^T$ be the weighted vector with $\partial_j\in[0,1]$ and $\sum_{j=1}^{n}\partial_j=1.$

3. Some induced Einstein hybrid aggregation operators

Definition 8: The induced intuitionistic fuzzy Einstein hybrid averaging operator can be stated as:

$$\text{I-IFEHA}_{\partial,\Im}\left(\left\langle 1,\Re_1\right\rangle,...,\left\langle n,\Re_n\right\rangle\right)=\left\langle\frac{\prod_{j=1}^{n}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}-\prod_{j=1}^{n}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{n}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{n}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}},\frac{2\prod_{j=1}^{n}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{n}\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{n}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}\right\rangle \tag{9}$$

where $\dot{\Re}_{\delta(j)}$ is the weighted intuitionistic fuzzy value $\dot{\Re}_j\left(\dot{\Re}_j=n\partial_j\Re_j, j=1,...,n\right)$ of the intuitionistic fuzzy ordered weighted averaging (IFOWA) pair $\left\langle u_j,\Re_j\right\rangle$ having the jth largest u_j,u_j in $\left\langle u_j,\Re_j\right\rangle$ is the order inducing variable and $\Re_j$ is the intuitionistic fuzzy argument variable, $\Im=\left(\Im_1,...,\Im_n\right)^T$ be the associated vector of I-IFEHA operator with $\Im_j\in[0,1]$ and $\sum_{j=1}^{n}\Im_j=1.$ $\partial=\left(\partial_1,...,\partial_n\right)^T$ be the weighted vector with $\partial_j\in[0,1]$ and $\sum_{j=1}^{n}\partial_j=1.$

Theorem 1: Let $\left\langle u_j,\Re_j\right\rangle(j=1,2,3)$ be a family of 2-tuples, and $\sigma>0,$ then

1. $\Re_1\oplus\Re_2=\Re_2\oplus\Re_1$
2. $\Re_1\otimes\Re_2=\Re_2\otimes\Re_1$
3. $\left(\Re_1\oplus\Re_2\right)\oplus\Re_3=\Re_1\oplus\left(\Re_2\oplus\Re_3\right)$

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

4. $(\mathfrak{R}_1 \otimes \mathfrak{R}_2) \otimes \mathfrak{R}_3 = \mathfrak{R}_1 \otimes (\mathfrak{R}_2 \otimes \mathfrak{R}_3)$

5. $\sigma(\mathfrak{R}_1 \oplus \mathfrak{R}_2) = \sigma\mathfrak{R}_1 \oplus \sigma\mathfrak{R}_2$

6. $(\mathfrak{R}_1 \otimes \mathfrak{R}_2)^{\sigma} = (\mathfrak{R}_1)^{\sigma} \otimes (\mathfrak{R}_2)^{\sigma}$

Theorem 2: Let $\langle u_j, \mathfrak{R}_j \rangle (j = 1,2,3)$ be a family of 2-tuples, and σ, then

1. $(\mathfrak{R}_1)^c \cup (\mathfrak{R}_2)^c = (\mathfrak{R}_1 \cap \mathfrak{R}_2)^c$

2. $(\mathfrak{R}_1)^c \cap (\mathfrak{R}_2)^c = (\mathfrak{R}_1 \cup \mathfrak{R}_2)^c$

3. $(\mathfrak{R}_1 \cup \mathfrak{R}_2) \cap \mathfrak{R}_2 = \mathfrak{R}_2$

4. $(\mathfrak{R}_1 \cap \mathfrak{R}_2) \cup \mathfrak{R}_2 = \mathfrak{R}_2$

5. $\left(\mathfrak{R}^c\right)^{\sigma} = (\sigma\mathfrak{R})^c$

6. $\sigma\left(\mathfrak{R}^c\right) = \left(\mathfrak{R}^{\sigma}\right)^c$

Theorem 3: Let $\langle u_j, \mathfrak{R}_j \rangle (j = 1,2,3)$ be a family of 2-tuples, then

1. $(\mathfrak{R}_1 \cup \mathfrak{R}_2) \oplus (\mathfrak{R}_1 \cap \mathfrak{R}_2) = \mathfrak{R}_1 \oplus \mathfrak{R}_2$

2. $(\mathfrak{R}_1 \cup \mathfrak{R}_2) \otimes (\mathfrak{R}_1 \cap \mathfrak{R}_2) = \mathfrak{R}_1 \otimes \mathfrak{R}_2$

3. $(\mathfrak{R}_1 \cup \mathfrak{R}_2) \cap \mathfrak{R}_3 = (\mathfrak{R}_1 \cap \mathfrak{R}_3) \cup (\mathfrak{R}_2 \cap \mathfrak{R}_3)$

4. $(\mathfrak{R}_1 \cap \mathfrak{R}_2) \cup \mathfrak{R}_3 = (\mathfrak{R}_1 \cup \mathfrak{R}_3) \cap (\mathfrak{R}_2 \cup \mathfrak{R}_3)$

5. $(\mathfrak{R}_1 \cup \mathfrak{R}_2) \oplus \mathfrak{R}_3 = (\mathfrak{R}_1 \oplus \mathfrak{R}_3) \cup (\mathfrak{R}_2 \oplus \mathfrak{R}_3)$

6. $(\mathfrak{R}_1 \cap \mathfrak{R}_2) \oplus \mathfrak{R}_3 = (\mathfrak{R}_1 \oplus \mathfrak{R}_3) \cap (\mathfrak{R}_2 \oplus \mathfrak{R}_3)$

Theorem 4: Let $\langle u_j, \mathfrak{R}_j \rangle (j = 1,2,...,n)$ be a family of 2-tuples, then their aggregated value by using the induced intuitionistic fuzzy Einstein hybrid averaging (I-IFEHA) operator is also intuitionistic fuzzy value, and

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

$$\text{I-IFEHA}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\ldots,\langle u_n,\Re_n\rangle\right)=\left\langle \frac{\prod_{j=1}^{n}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}-\prod_{j=1}^{n}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{n}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{n}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}},\frac{2\prod_{j=1}^{n}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{n}\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{n}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}\right\rangle \quad (10) \tag{10}$$

Proof: By mathematical induction. For $n=2$

$$\Im_1\langle u_1,\Re_1\rangle=\left\langle \frac{\left(1+\xi_{\dot{\Re}_1}\right)^{\Im_1}-\left(1-\xi_{\dot{\Re}_1}\right)^{\Im_1}}{\left(1+\xi_{\dot{\Re}_1}\right)^{\Im_1}+\left(1-\xi_{\dot{\Re}_1}\right)^{\Im_1}},\frac{2\left(\kappa_{\dot{\Re}_1}\right)^{\Im_1}}{\left(2-\kappa_{\dot{\Re}_1}\right)^{w_1}+\left(\kappa_{\dot{\Re}_1}\right)^{\Im_1}}\right\rangle$$

$$\Im_2\langle u_2,\Re_2\rangle=\left\langle \frac{\left(1+\xi_{\dot{\Re}_2}\right)^{\Im_2}-\left(1-\xi_{\dot{\Re}_2}\right)^{\Im_2}}{\left(1+\xi_{\dot{\Re}_2}\right)^{\Im_2}+\left(1-\xi_{\dot{\Re}_2}\right)^{\Im_2}},\frac{2\left(\kappa_{\dot{\Re}_2}\right)^{\Im_2}}{\left(2-\kappa_{\dot{\Re}_2}\right)^{\Im_2}+\left(\kappa_{\dot{\Re}_2}\right)^{\Im_2}}\right\rangle$$

Thus

$$\text{I-IFEHA}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\langle u_2,\Re_2\rangle\right)=\left\langle \frac{\prod_{j=1}^{2}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}-\prod_{j=1}^{2}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{2}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{2}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}},\frac{2\prod_{j=1}^{2}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{2}\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{2}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}\right\rangle$$

The given result is true for $n=2$, so it is true for $n=k$.

$$\text{I-IFEHA}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\ldots,\langle u_k,\Re_k\rangle\right)=\left\langle \frac{\prod_{j=1}^{k}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}-\prod_{j=1}^{k}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{k}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{k}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}},\frac{2\prod_{j=1}^{k}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{k}\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{k}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}\right\rangle$$

If the given result is true for $n=k$, then it is true for $n=k+1$

$$\text{I-IFEHA}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\ldots,\langle u_{k+1},\Re_{k+1}\rangle\right)=\left\langle \frac{\prod_{j=1}^{k}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}-\prod_{j=1}^{k}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{k}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{k}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}},\frac{2\prod_{j=1}^{k}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{k}\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{k}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}\right\rangle$$

$$\oplus_{\varepsilon}\left\langle \frac{\left(1+\xi_{\dot{\Re}_{k+1}}\right)^{\Im_{k+1}}-\left(1-\xi_{\dot{\Re}_{k+1}}\right)^{\Im_{k+1}}}{\left(1+\xi_{\dot{\Re}_{k+1}}\right)^{\Im_{k+1}}+\left(1-\xi_{\dot{\Re}_{k+1}}\right)^{\Im_{k+1}}},\frac{2\left(\kappa_{\dot{\Re}_{k+1}}\right)^{\Im_{k+1}}}{\left(2-\kappa_{\dot{\Re}_{k+1}}\right)^{\Im_{k+1}}+\left(\kappa_{\dot{\Re}_{k+1}}\right)^{\Im_{k+1}}}\right\rangle \quad (11) \tag{11}$$

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

Let

$$p_1=\prod_{j=1}^{k}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}-\prod_{j=1}^{k}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j},\ p_2=\left(1+\xi_{\dot{\Re}_{k+1}}\right)^{\Im_{k+1}}-\left(1-\xi_{\dot{\Re}_{k+1}}\right)^{\Im_{k+1}},\ r_1=2\prod_{j=1}^{k}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}$$

$$q_1=\prod_{j=1}^{k}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{k}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j},\ q_2=\left(1+\xi_{\dot{\Re}_{k+1}}\right)^{\Im_{k+1}}+\left(1-\xi_{\dot{\Re}\xi_{k+1}}\right)^{\Im_{k+1}},\ r_2=2\left(\kappa_{\dot{\Re}_{k+1}}\right)^{\Im_{k+1}}$$

and $s_1=\prod_{j=1}^{k}\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{k}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j},\ s_2=\left(2-\kappa_{\dot{\Re}_{k+1}}\right)^{\Im_{k+1}}+\left(\kappa_{\dot{\Re}_{k+1}}\right)^{\Im_{k+1}}$.

Now putting these values in equation (11), we have

$$\begin{aligned}\text{I-IFEHA}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\ldots,\langle u_{k+1},\Re_{k+1}\rangle\right)&=\left\langle\frac{p_1}{q_1},\frac{r_1}{s_1}\right\rangle\oplus_{\varepsilon}\left\langle\frac{p_2}{q_2},\frac{r_2}{s_2}\right\rangle\\&=\left\langle\frac{\left(\frac{p_1}{q_1}\right)+\left(\frac{p_2}{q_2}\right)}{1+\left(\frac{p_1}{q_1}\right)\left(\frac{p_2}{q_2}\right)},\frac{\frac{r_1}{s_1}\frac{r_2}{s_2}}{1+\left(1-\left(\frac{r_1}{s_1}\right)\right)\left(1-\left(\frac{r_2}{s_2}\right)\right)}\right\rangle\\&=\left\langle\frac{p_1q_2+p_2q_1}{q_1q_2+p_1p_2},\frac{r_1r_2}{2s_1s_2-s_1r_2-r_1s_2+r_1r_2}\right\rangle\end{aligned}\tag{12}$$

Putting the values of $p_1q_2+p_2q_1, q_1q_2+p_1p_2, r_1r_2, 2s_1s_2-s_1r_2-r_1s_2+r_1r_2$ in equation (12), we have

$$\text{I-IFEHA}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\ldots,\langle u_{k+1},\Re_{k+1}\rangle\right)=\left\langle\frac{\prod_{j=1}^{k+1}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}-\prod_{j=1}^{k+1}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{k+1}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{k+1}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}},\frac{2\prod_{j=1}^{k+1}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{k+1}\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{k+1}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}\right\rangle$$

Thus the given result is true for $n=k+1$. Thus equation (10), true for all n.

Lemma 1: [31] Let $\Re_j\succ 0, \Im_j\succ 0\left(j=1,2,\ldots,n\right)$ and $\sum_{j=1}^{n}\Im_j=1.$ then

$$\prod_{j=1}^{n}\left(\Re_j\right)^{\Im_j}\le\sum_{j=1}^{n}\Im_j\Re_j\tag{13}$$

where the equality holds if and only if $\Re_j\left(j=1,2,\ldots,n\right)=\Re$

Theorem 5: Let $\left\langle u_j,\Re_j\right\rangle\left(j=1,2,\ldots,n\right)$ be a family of 2-tuples, then

$$\text{I-IFEHA}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\langle u_2,\Re_2\rangle,\ldots,\langle u_n,\Re_n\rangle\right)\le \text{I-IFHA}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\langle u_2,\Re_2\rangle,\ldots,\langle u_n,\Re_n\rangle\right) \quad (14)$$

Proof: As

$$\prod_{j=1}^{n}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{n}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}\le\sum_{j=1}^{n}\Im_j\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)+\sum_{j=1}^{n}\Im_j\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)$$

Now

$$\sum_{j=1}^{n}\Im_j\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)+\sum_{j=1}^{n}\Im_j\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)=2$$

Hence

$$\prod_{j=1}^{n}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{n}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}\le 2$$

Also

$$\frac{\prod_{j=1}^{n}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}-\prod_{j=1}^{n}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{n}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{n}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}=1-\frac{2\prod_{j=1}^{n}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{n}\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{n}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}\le 1-\prod_{j=1}^{n}\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j} \quad (15)$$

If $\xi_{\dot{\Re}_{\delta(j)}}$ are equal for all n, then the equality holds. Again

$$\prod_{j=1}^{n}\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{n}\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}\le\sum_{j=1}^{n}\Im_j\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)+\sum_{j=1}^{n}\Im_j\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)$$

As

The linked image cannot be displayed. The file may have been moved, renamed, or deleted. Verify that the link points to the correct file and location.

So

The linked image cannot be displayed. The file may have been moved, renamed, or deleted. Verify that the link points to the correct file and location.

Thus

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

[illegible] (16)

If [illegible] are equal for all n, then the equality holds. Let

[illegible] (17)

And

[illegible] (18)

Then equations (15) and (16) can be transformed into the following forms:

[illegible] and [illegible], then [illegible]. Thus [illegible]

There are two cases in this direction.

Case 1: If [illegible], then by Definition 4 we have

[illegible] (19)

Case 2: If [illegible], then we have [illegible] and $\kappa_{\dot{\mathfrak{R}}} = \kappa_{\dot{\mathfrak{R}}^{\varepsilon}}$ this implies that $\xi_{\dot{\mathfrak{R}}} + \kappa_{\dot{\mathfrak{R}}} = \xi_{\dot{\mathfrak{R}}^{\varepsilon}} + \kappa_{\dot{\mathfrak{R}}^{\varepsilon}}$ hence $H\left(\dot{\mathfrak{R}}\right) = H\left(\dot{\mathfrak{R}}^{\varepsilon}\right)$. Thus by Definition 4 we have

$$\text{I-IFEHA}_{\partial,\mathfrak{I}}\left(\langle u_1,\mathfrak{R}_1\rangle,\langle u_2,\mathfrak{R}_2\rangle,\ldots,\langle u_n,\mathfrak{R}_n\rangle\right) = \text{I-IFHA}_{\partial,\mathfrak{I}}\left(\langle u_1,\mathfrak{R}_1\rangle,\langle u_2,\mathfrak{R}_2\rangle,\ldots,\langle u_n,\mathfrak{R}_n\rangle\right) \quad (20)$$

Hence from equations (19) and (20), we have

$$\text{I-IFEHA}_{\partial,\mathfrak{I}}\left(\langle u_1,\mathfrak{R}_1\rangle,\langle u_2,\mathfrak{R}_2\rangle,\ldots,\langle u_n,\mathfrak{R}_n\rangle\right) \le \text{I-IFHA}_{\partial,\mathfrak{I}}\left(\langle u_1,\mathfrak{R}_1\rangle,\langle u_2,\mathfrak{R}_2\rangle,\ldots,\langle u_n,\mathfrak{R}_n\rangle\right)$$

The proof is completed.

Example 1: Let

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

$\langle u_1, \Re_1 \rangle = \langle 0.30, (0.70, 0.20) \rangle, \langle u_2, \Re_2 \rangle = \langle 0.80, (0.60, 0.30) \rangle, \langle u_3, \Re_3 \rangle = \langle 0.70, (0.50, 0.50) \rangle$ and $\langle u_4, \Re_4 \rangle = \langle 0.50, (0.40, 0.60) \rangle$, , , then

By ordering with respect to the first element, then we have

$$\langle u_2, \Re_2 \rangle = \langle 0.80, (0.60, 0.30) \rangle, \langle u_3, \Re_3 \rangle = \langle 0.70, (0.50, 0.50) \rangle, \langle u_4, \Re_4 \rangle = \langle 0.50, (0.40, 0.60) \rangle, \langle u_1, \Re_1 \rangle = \langle 0.30, (0.70, 0.20) \rangle$$

Hence

Again using the I-IFEHA operator, then we have

Hence

Thus

Theorem 6: Induced intuitionistic fuzzy Einstein hybrid averaging operator satiates the following properties.

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

1. **Idempotency:** If for all j, then

(21)

Proof: As for all j, then we have

2. **Boundedness:** Le ,

then

(22)

Proof: From the above conditions, we have , this implies that

then we have

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

Again i.e., , then we have

Let

(25)

Then equations (23), (24) can be written as: and

. Then

and

Now there are three conditions.

1. If and . Then by Definition 4, we have

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

[illegible] (26)

2. If [illegible] then [illegible], this implies that [illegible] and $\kappa_{\dot{\mathfrak{R}}} = \kappa_{\dot{\mathfrak{R}}_{\max}}$ hence $H(\dot{\mathfrak{R}}) = \xi_{\dot{\mathfrak{R}}} + \kappa_{\dot{\mathfrak{R}}} = \xi_{\dot{\mathfrak{R}}_{\max}} + \kappa_{\dot{\mathfrak{R}}_{\max}} = H(\dot{\mathfrak{R}}_{\max})$. By Definition 4, we have

$$\text{I-IFEHA}_{\partial,\mathfrak{I}}\left(\langle u_1,\mathfrak{R}_1\rangle,\langle u_2,\mathfrak{R}_2\rangle,...,\langle u_n,\mathfrak{R}_n\rangle\right) = \dot{\mathfrak{R}}_{\max} \tag{27}$$

3. If $S(\dot{\mathfrak{R}}) = S(\dot{\mathfrak{R}}_{\min})$, then, $\xi_{\dot{\mathfrak{R}}} - \kappa_{\dot{\mathfrak{R}}} = \xi_{\dot{\mathfrak{R}}_{\min}} - \kappa_{\dot{\mathfrak{R}}_{\max}}$, this implies that $\xi_{\dot{\mathfrak{R}}} = \xi_{\dot{\mathfrak{R}}_{\min}}$ and $\kappa_{\dot{\mathfrak{R}}} = \kappa_{\dot{\mathfrak{R}}_{\min}}$ henc $H(\dot{\mathfrak{R}}) = \xi_{\dot{\mathfrak{R}}} + \kappa_{\dot{\mathfrak{R}}} = \xi_{\dot{\mathfrak{R}}_{\min}} + \kappa_{\dot{\mathfrak{R}}_{\min}} = H(\dot{\mathfrak{R}}_{\min})$. By Definition 4, we have

$$\text{I-IFEHA}_{\partial,\mathfrak{I}}\left(\langle u_1,\mathfrak{R}_1\rangle,\langle u_2,\mathfrak{R}_2\rangle,...,\langle u_n,\mathfrak{R}_n\rangle\right) = \dot{\mathfrak{R}}_{\min} \tag{28}$$

Thus from equations (26) to (28) equation (22) always holds.

4. **Monotonicity:** If $\left\langle u_j,\mathfrak{R}_j^*\right\rangle (j=1,2,...,n)$ be a family 2-tuples, where $\dot{\mathfrak{R}}_j \le \dot{\mathfrak{R}}_j^*$, then

$$\text{I-IFEHA}_{\partial,\mathfrak{I}}\left(\langle u_1,\mathfrak{R}_1\rangle,\langle u_2,\mathfrak{R}_2\rangle...,\langle u_n,\mathfrak{R}_n\rangle\right) \le \text{I-IFEHA}_{\partial,\mathfrak{I}}\left(\left\langle u_1,\mathfrak{R}_1^*\right\rangle,\left\langle u_2,\mathfrak{R}_2^*\right\rangle,...,\left\langle u_n,\mathfrak{R}_n^*\right\rangle\right) \tag{29}$$

Proof: The proof is similar to above.

Theorem 7: Intuitionistic fuzzy Einstein weighted averaging operator is a specific case of the induced intuitionistic fuzzy Einstein hybrid averaging operator.

Proof: Let $\mathfrak{I} = \left(\frac{1}{n},\frac{1}{n},...,\frac{1}{n},\right)^T$, then we have

$$\begin{aligned}\text{I-IFEHA}_{\partial,\mathfrak{I}}\left(\langle u_1,\mathfrak{R}_1\rangle,\langle u_2,\mathfrak{R}_2\rangle,...,\langle u_n,\mathfrak{R}_n\rangle\right) &= \mathfrak{I}_1\dot{\mathfrak{R}}_{\delta(1)} \oplus_\varepsilon \mathfrak{I}_2\dot{\mathfrak{R}}_{\delta(2)} \oplus_\varepsilon ... \oplus_\varepsilon \mathfrak{I}_n\dot{\mathfrak{R}}_{\delta(n)} \\ &= \frac{1}{n}\left(\dot{\mathfrak{R}}_{\delta(1)}\right) \oplus_\varepsilon \frac{1}{n}\left(\dot{\mathfrak{R}}_{\delta(2)}\right) \oplus_\varepsilon ... \oplus_\varepsilon \frac{1}{n}\left(\dot{\mathfrak{R}}_{\delta(n)}\right). \\ &= \text{IFEWA}_\partial\left(\mathfrak{R}_1,\mathfrak{R}_2,...,\mathfrak{R}_n\right)\end{aligned}$$

Theorem 8: Induced intuitionistic fuzzy Einstein ordered weighted averaging operator is a specific case of the induced intuitionistic fuzzy Einstein hybrid averaging operator.

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

Proof: Let $\partial = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n},\right)^T$, $\dot{\Re}_{\delta(j)} = n\partial_j \Re_{\delta(j)} = \Re_{\delta(j)}$, then

$$\begin{aligned}\text{I-IFEHA}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\langle u_2,\Re_2\rangle,...,\langle u_n,\Re_n\rangle\right) &= \Im_1\dot{\Re}_{\delta(1)} \oplus_\varepsilon \Im_2\dot{\Re}_{\delta(2)} \oplus_\varepsilon ... \oplus_\varepsilon \Im_n\dot{\Re}_{\delta(n)} \\ &= \Im_1\Re_{\delta(1)} \oplus_\varepsilon \Im_2\Re_{\delta(2)} \oplus_\varepsilon ... \oplus_\varepsilon \Im_n\Re_{\delta a(n)} \\ &= \text{I-IFEOWA}_\partial\left(\langle u_1,\Re_1\rangle,\langle u_2,\Re_2\rangle,...,\langle u_n,\Re_n\rangle\right)\end{aligned}$$

Definition 9: An induced intuitionistic fuzzy Einstein hybrid geometric operator can be defined as:

$$\text{I-IFEHG}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\langle u_2,\Re_2\rangle,...,\langle u_n,\Re_n\rangle\right) = \left\langle \frac{2\prod_{j=1}^{n}\left(\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{n}\left(2-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{n}\left(\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}, \frac{\prod_{j=1}^{n}\left(1+\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}-\prod_{j=1}^{n}\left(1-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{n}\left(1+\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{n}\left(1-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}} \right\rangle \quad (30)$$

Where $\dot{\Re}_{\delta(j)}$ is the weighted intuitionistic fuzzy value $\dot{\Re}_j\left(\dot{\Re}_j = \left(\Re_j\right)^{n\partial_j}, j = 1,2,...,n\right)$ of the intuitionistic fuzzy ordered weighted geometric (IFOWG) pair $\langle u_j,\Re_j\rangle$ having the jth largest u_j in $\langle u_j,\Re_j\rangle$ is the order inducing variable and $\Re_j$ is the intuitionistic fuzzy argument variable, $\Im = \left(\Im_1,\Im_2,...,\Im_n\right)^T$ be the weighted vector of I-IFEHG operator with $\Im_j \in [0,1]$ and $\sum_{j=1}^{n}\Im_j = 1$. $\partial = \left(\partial_1,\partial_2,...,\partial_n\right)^T$ with $\partial_j \in [0,1]$ and $\sum_{j=1}^{n}\partial_j = 1$.

Theorem 9: Let $\langle u_j,\Re_j\rangle (j = 1,2,...,n)$ be a family of 2-tuples, then their aggregated value by using the I-IFEHG is also a IFV and

$$\text{I-IFEHG}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\langle u_2,\Re_2\rangle,...,\langle u_n,\Re_n\rangle\right) = \left\langle \frac{2\prod_{j=1}^{n}\left(\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{n}\left(2-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{n}\left(\xi_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}, \frac{\prod_{j=1}^{n}\left(1+\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}-\prod_{j=1}^{n}\left(1-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}}{\prod_{j=1}^{n}\left(1+\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}+\prod_{j=1}^{n}\left(1-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\Im_j}} \right\rangle \quad (31)$$

Proof: The proof follows from Theorem 4.

Theorem 10: Let $\langle u_j,\Re_j\rangle (j = 1,2,...,n)$ be a family of 2-tuples, then

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

$$\text{I-IFHG}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\langle u_2,\Re_2\rangle,...,\langle u_n,\Re_n\rangle\right)\le \text{I-IFEHG}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\langle u_2,\Re_2\rangle,...,\langle u_n,\Re_n\rangle\right) \quad (32)$$

Proof: The proof follows from Theorem 5.

Example 2:

$\Re_1=\langle 0.80,(0.70,0.20)\rangle,\Re_2=\langle 0.50,(0.60,0.30)\rangle,\Re_3=\langle 0.40,(0.50,0.50)\rangle,\Re_4=\langle 0.30,(0.40,0.60)\rangle$ are four values and $\partial=(0.10,0.20,0.30,0.40)^T$, $\Im=(0.10,0.20,0.30,0.40)^T$, then we have $\dot{\Re}_1=(0.867,0.085),\dot{\Re}_2=(0.664,0.248),\dot{\Re}_3=(0.435,0.564),\dot{\Re}_4=(0.230,0.769)$. Hence

$$\left\langle u_{\delta(1)},\dot{\Re}_{\delta(1)}\right\rangle=(0.867,0.085),\left\langle u_{\delta(2)},\dot{\Re}_{\delta(2)}\right\rangle=(0.664,0.248),\left\langle u_{\delta(3)},\dot{\Re}_{\delta(3)}\right\rangle=(0.435,0.564),\left\langle u_{\delta(4)},\dot{\Re}_{\delta(4)}\right\rangle=(0.230,0.769)$$

$$\begin{aligned}&\text{I-IFHG}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\langle u_2,\Re_2\rangle,\langle u_3,\Re_3\rangle,\langle u_4,\Re_4\rangle\right)\\&=\left\langle (0.867)^{0.10}(0.664)^{0.20}(0.435)^{0.30}(0.230)^{0.40},1-(1-0.085)^{0.10}(1-0.248)^{0.20}(1-0.564)^{0.30}(1-0.769)^{0.40}\right\rangle\\&=\left\langle (0.867)^{0.10}(0.664)^{0.20}(0.435)^{0.30}(0.230)^{0.40},1-(0.915)^{0.10}(0.752)^{0.20}(0.436)^{0.30}(0.231)^{0.40}\right\rangle=\langle 0.393,0.594\rangle\end{aligned}$$

Again using the I-IFEHG geometric operator, then we have

$\dot{\Re}_1=(0.876,0.080),\dot{\Re}_2=(0.673,0.242),\dot{\Re}_3=(0.422,0.577),\dot{\Re}_4=(0.196,0.803)$. Hence

$$\left\langle u_{\delta(1)},\dot{\Re}_{\delta(1)}\right\rangle=(0.876,0.080),\left\langle u_{\delta(2)},\dot{\Re}_{\delta(2)}\right\rangle=(0.673,0.242),\left\langle u_{\delta(3)},\dot{\Re}_{\delta(3)}\right\rangle=(0.422,0.577),\left\langle u_{\delta(4)},\dot{\Re}_{\delta(4)}\right\rangle=(0.196,0.803)$$

Thus

$$\begin{aligned}&\text{I-IFEHWG}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\langle u_2,\Re_2\rangle,\langle u_3,\Re_3\rangle,\langle u_4,\Re_4\rangle\right)\\&=\left\langle \begin{array}{l}\dfrac{2(0.876)^{0.10}(0.673)^{0.20}(0.422)^{0.30}(0.196)^{0.40}}{(2-0.876)^{0.10}(2-0.673)^{0.20}(2-0.422)^{0.30}(2-0.196)^{0.40}+(0.876)^{0.10}(0.673)^{0.20}(0.422)^{0.30}(0.196)^{0.40}},\\ \dfrac{(1+0.080)^{0.10}(1+0.242)^{0.20}(1+0.577)^{0.30}(1+0.803)^{0.40}-(1-0.080)^{0.10}(1-0.242)^{0.20}(1-0.577)^{0.30}(1-0.803)^{0.40}}{(1+0.080)^{0.10}(1+0.242)^{0.20}(1+0.577)^{0.30}(1+0.803)^{0.40}+(1-0.080)^{0.10}(1-0.242)^{0.20}(1-0.577)^{0.30}(1-0.803)^{0.40}}\end{array}\right\rangle\\&=\left\langle \begin{array}{l}\dfrac{2(0.876)^{0.10}(0.673)^{0.20}(0.422)^{0.30}(0.196)^{0.40}}{(1.124)^{0.10}(1.327)^{0.20}(1.578)^{0.30}(1.804)^{0.40}+(0.876)^{0.10}(0.673)^{0.20}(0.422)^{0.30}(0.196)^{0.40}},\\ \dfrac{(1.080)^{0.10}(1.242)^{0.20}(1.577)^{0.30}(1.803)^{0.40}-(0.920)^{0.10}(0.758)^{0.20}(0.423)^{0.30}(0.197)^{0.40}}{(1.080)^{0.10}(1.242)^{0.20}(1.577)^{0.30}(1.803)^{0.40}+(0.920)^{0.10}(0.758)^{0.20}(0.423)^{0.30}(0.197)^{0.40}}\end{array}\right\rangle=\langle 0.404,0.585\rangle\end{aligned}$$

Theorem 11: Intuitionistic fuzzy Einstein weighted geometric operator is a specific case of the induced intuitionistic fuzzy Einstein hybrid geometric operator.

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

Proof: Let $\Im = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$, then we have

$$\begin{aligned}\text{I-IFEHG}_{\partial,\Im}\left(\langle u_1, \Re_1\rangle, \langle u_2, \Re_2\rangle, ..., \langle u_n, \Re_n\rangle\right) &= \left(\dot{\Re}_{\delta(1)}\right)^{\Im_1} \otimes_\varepsilon \left(\dot{\Re}_{\delta(2)}\right)^{\Im_2} \otimes_\varepsilon ... \otimes_\varepsilon \left(\dot{\Re}_{\delta(n)}\right)^{\Im_n} \\ &= \left(\dot{\Re}_{\delta(1)}\right)^{\frac{1}{n}} \otimes_\varepsilon \left(\dot{\Re}_{\delta(2)}\right)^{\frac{1}{n}} \otimes_\varepsilon ... \otimes_\varepsilon \left(\dot{\Re}_{\delta(n)}\right)^{\frac{1}{n}} \\ &= \text{IFEWG}_\partial\left(\Re_1, \Re_2, ..., \Re_n\right)\end{aligned}$$

Theorem 12: Induced intuitionistic fuzzy Einstein ordered weighted geometric operator is a specific case of the induced intuitionistic fuzzy Einstein hybrid geometric operator

Proof: Let $\partial = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$ and $\dot{\Re}_{\delta(j)} = \left(\Re_{\delta(j)}\right)^{n\partial_j} = \Re_{\delta(j)}$, then

$$\begin{aligned}\text{I-IFEHG}_{\partial,\Im}\left(\langle u_1, \Re_1\rangle, \langle u_2, \Re_2\rangle, ..., \langle u_n, \Re_n\rangle\right) &= \left(\dot{\Re}_{\delta(1)}\right)^{\Im_1} \otimes_\varepsilon \left(\dot{\Re}_{\delta(2)}\right)^{\Im_2} \otimes_\varepsilon ... \otimes_\varepsilon \left(\dot{\Re}_{\delta(n)}\right)^{\Im_n} \\ &= \text{I-IFEOWG}_\Im\left(\langle u_1, \Re_1\rangle, \langle u_2, \Re_2\rangle, ..., \langle u_n, \Re_n\rangle\right)\end{aligned}$$

Definition 10: The Induced generalized intuitionistic fuzzy Einstein hybrid geometric (I-GIFEHG) operator can be defined as:

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

$$\text{I-GIFEHG}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\langle u_2,\Re_2\rangle,\langle u_3,\Re_3\rangle,\ldots,\langle u_n,\Re_n\rangle\right)$$

$$=\left\langle \begin{array}{l} \dfrac{\left(\prod_{j=1}^{n}\left\{\left(1+\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}+3\left(1-\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}\right\}^{\Im_j}+3\prod_{j=1}^{n}\left\{\left(1+\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}-\left(1-\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}-\left(\prod_{j=1}^{n}\left\{\left(1+\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}+3\left(1-\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}\right\}^{\Im_j}-\prod_{j=1}^{n}\left\{\left(1+\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}-\left(1-\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}}{\left(\prod_{j=1}^{n}\left\{\left(1+\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}+3\left(1-\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}\right\}^{\Im_j}+3\prod_{j=1}^{n}\left\{\left(1+\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}-\left(1-\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}+\left(\prod_{j=1}^{n}\left\{\left(1+\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}+3\left(1-\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}\right\}^{\Im_j}-\prod_{j=1}^{n}\left\{\left(1+\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}-\left(1-\xi\dot{\Re}_{\delta(j)}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}}, \\ \dfrac{2\left(\prod_{j=1}^{n}\left\{\left(2-\kappa\dot{\Re}_{\delta(j)}\right)^{\sigma}+3\left(\kappa\dot{\Re}_{\delta(j)}\right)^{\sigma}\right\}^{\Im_j}-\prod_{j=1}^{n}\left\{\left(2-\kappa\dot{\Re}_{\delta(j)}\right)^{\sigma}-\left(\kappa\dot{\Re}_{\delta(j)}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}}{\left(\prod_{j=1}^{n}\left\{\left(2-\kappa\dot{\Re}_{\delta(j)}\right)^{\sigma}+3\left(\kappa\dot{\Re}_{\delta(j)}\right)^{\sigma}\right\}^{\Im_j}+3\prod_{j=1}^{n}\left\{\left(2-\kappa\dot{\Re}_{\delta(j)}\right)^{\sigma}-\left(\kappa\dot{\Re}_{\delta(j)}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}+\left(\prod_{j=1}^{n}\left\{\left(2-\kappa\dot{\Re}_{\delta(j)}\right)^{\sigma}+3\left(\kappa\dot{\Re}_{\delta(j)}\right)^{\sigma}\right\}^{\Im_j}-\prod_{j=1}^{n}\left\{\left(2-\kappa\dot{\Re}_{\delta(j)}\right)^{\sigma}-\left(\kappa\dot{\Re}_{\delta(j)}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}} \end{array}\right\rangle \quad (33)$$

Where $\dot{\Re}_{\delta(j)}$ is the weighted intuitionistic fuzzy value $\dot{\Re}_j\left(\dot{\Re}_j=\left(\Re_j\right)^{n\partial_j},\ j=1,2,\ldots,n\right)$ of the intuitionistic fuzzy ordered weighted geometric (IFOWG) pair $\langle u_j,\Re_j\rangle$ having the jth largest u_j in $\langle u_j,\Re_j\rangle$, $\Im=\left(\Im_1,\Im_2,\ldots,\Im_n\right)^T$ be the weighted vector of I-GIFEHG operator with $\Im_j\in[0,1]$ and $\sum_{j=1}^{n}\Im_j=1$. $\partial=\left(\partial_1,\partial_2,\ldots,\partial_n\right)^T$ with $\partial_j\in[0,1]$ and $\sum_{j=1}^{n}\partial_j=1$. And σ is a real number greater than zero.

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

Theorem 13: Let $\langle u_j, \mathfrak{R}_j \rangle (j = 1,2,...,n)$ be a family of 2-tuples, then their aggregated IFV by using the I-GIFEHG operator can be expressed as:

$$\text{I-GIFEHG}_{\partial,\mathfrak{I}}\left(\langle u_1,\mathfrak{R}_1\rangle,\langle u_2,\mathfrak{R}_2\rangle,\langle u_3,\mathfrak{R}_3\rangle,...,\langle u_n,\mathfrak{R}_n\rangle\right)$$

$$= \left\langle \begin{array}{l} \dfrac{\left(\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}+3\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}\right)^{\frac{1}{\sigma}} - \left(\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}-\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}\right)^{\frac{1}{\sigma}}}{\left(\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}+3\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}\right)^{\frac{1}{\sigma}} + \left(\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}-\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}\right)^{\frac{1}{\sigma}}}, \\ \dfrac{2\left(\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}-\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}\right)^{\frac{1}{\sigma}}}{\left(\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}+3\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}\right)^{\frac{1}{\sigma}} + \left(\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}-\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}\right)^{\frac{1}{\sigma}} \end{array} \right\rangle \quad (34)$$

Proof: Since

$$\sigma \cdot_{\varepsilon} \dot{\mathfrak{R}} = \left\langle \frac{\left(1+\xi_{\dot{\mathfrak{R}}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}}\right)^{\sigma}}{\left(1+\xi_{\dot{\mathfrak{R}}}\right)^{\sigma}+\left(1-\xi_{\dot{\mathfrak{R}}}\right)^{\sigma}}, \frac{2\left(\kappa_{\dot{\mathfrak{R}}}\right)^{\sigma}}{\left(2-\kappa_{\dot{\mathfrak{R}}}\right)^{\sigma}+\left(\kappa_{\dot{\mathfrak{R}}}\right)^{\sigma}} \right\rangle$$

Then

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

$$\left(\bigotimes_{j=1}^{n}{}^{\sigma} j\,\varepsilon\left(\dot{\mathfrak{R}}_{\delta(j)}\right)\right)^{\mathfrak{I}_j}$$

$$=\left\langle \frac{2\prod_{j=1}^{n}\left(\frac{\left(1+\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}-\left(1-\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}{\left(1+\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(1-\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}\right)^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left(2-\frac{\left(1+\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}-\left(1-\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}{\left(1+\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(1-\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}\right)^{\mathfrak{I}_j}+\prod_{j=1}^{n}\left(\frac{\left(1+\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}-\left(1-\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}{\left(1+\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(1-\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}\right)^{\mathfrak{I}_j}}, \frac{\prod_{j=1}^{n}\left(1+\frac{2\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}{\left(2-\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}\right)^{\mathfrak{I}_j}-\prod_{j=1}^{n}\left(1-\frac{2\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}{\left(2-\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}\right)^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left(1+\frac{2\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}{\left(2-\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}\right)^{\mathfrak{I}_j}+\prod_{j=1}^{n}\left(1-\frac{2\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}{\left(2-\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}\right)^{\mathfrak{I}_j}}\right\rangle$$

$$=\left\langle \frac{\left(\frac{2\prod_{j=1}^{n}\left(\left(1+\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}-\left(1-\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}\right)^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left(\left(1+\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(1-\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}\right)^{w_j}}\right)}{\prod_{j=1}^{n}\left(\frac{2\left(\left(1+\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(1-\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}\right)-\left(1+\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(1-\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}{\left(1+\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(1-\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}\right)^{\mathfrak{I}_j}+\left(\frac{\prod_{j=1}^{n}\left(\left(1+\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}-\left(1-\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}\right)^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left(\left(1+\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(1-\xi\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}\right)^{\mathfrak{I}_j}}\right)}, \frac{\prod_{j=1}^{n}\left(\frac{\left(2-\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+2\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}{\left(2-\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}\right)^{\mathfrak{I}_j}-\prod_{j=1}^{n}\left(\frac{\left(2-\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}-2\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}{\left(2-\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}\right)^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left(\frac{\left(2-\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+2\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}{\left(2-\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}\right)^{\mathfrak{I}_j}+\prod_{j=1}^{n}\left(\frac{\left(2-\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}-2\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}{\left(2-\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}+\left(\kappa\dot{\mathfrak{R}}_{\delta(j)}\right)^{\sigma}}\right)^{\mathfrak{I}_j}}\right\rangle$$

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

$$
=\left\langle \frac{\left(\dfrac{2\prod_{j=1}^{n}\left(\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right)^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left(\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right)^{\mathfrak{I}_j}}\right)}{\prod_{j=1}^{n}\left(\dfrac{2\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+2\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}\right)^{\mathfrak{I}_j}+\dfrac{\prod_{j=1}^{n}\left(\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right)^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left(\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right)^{\mathfrak{I}_j}}}, \frac{\prod_{j=1}^{n}\left(\dfrac{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}\right)^{\mathfrak{I}_j}-\prod_{j=1}^{n}\left(\dfrac{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}\right)^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left(\dfrac{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}\right)^{\mathfrak{I}_j}+\prod_{j=1}^{n}\left(\dfrac{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}\right)^{\mathfrak{I}_j}} \right\rangle
$$

$$
=\left\langle \frac{\left(\dfrac{2\left(\prod_{j=1}^{n}\left(\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right)\right)^{\mathfrak{I}_j}}{\left(\prod_{j=1}^{n}\left(\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right)\right)^{\mathfrak{I}_j}}\right)}{\left(\dfrac{\prod_{j=1}^{n}\left(\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right)^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left(\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right)^{\mathfrak{I}_j}}\right)+\left(\dfrac{\prod_{j=1}^{n}\left(\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right)^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left(\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right)^{\mathfrak{I}_j}}\right)}, \frac{\prod_{j=1}^{n}\left(\dfrac{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}\right)^{\mathfrak{I}_j}-\prod_{j=1}^{n}\left(\dfrac{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}\right)^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left(\dfrac{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}\right)^{\mathfrak{I}_j}+\prod_{j=1}^{n}\left(\dfrac{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}}\right)^{\mathfrak{I}_j}} \right\rangle
$$

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

$$=\left\langle \frac{2\prod_{j=1}^{n}\left(\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right)^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}+\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}, \frac{\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}-\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}+\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}} \right\rangle$$

Thus

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

$$\frac{1}{\sigma}\left(\bigotimes_{j=1}^{n}\sigma\cdot\varepsilon\left(\dot{\mathfrak{R}}_{\delta(j)}\right)^{\mathfrak{I}_j}\right)$$

$$=\left\langle\begin{array}{l}
\dfrac{\left(1+\dfrac{2\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}+\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}\right)^{\frac{1}{\sigma}}-\left(1-\dfrac{2\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}+\prod_{j=1}^{n}\left\{\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\sigma}-\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}\right)^{\frac{1}{\sigma}}}{\left(1+\dfrac{2\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}+\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}\right)^{\frac{1}{\sigma}}+\left(1-\dfrac{2\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}+\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}\right)^{\frac{1}{\sigma}}}, \\
\dfrac{2\left(\dfrac{\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}-\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}+\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}\right)^{\frac{1}{\sigma}}}{\left(2-\dfrac{\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}-\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}+\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}\right)^{\frac{1}{\sigma}}+\left(\dfrac{\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}-\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}{\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}+\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\mathfrak{R}}_{\delta(j)}}\right)^{\sigma}\right\}^{\mathfrak{I}_j}}\right)^{\frac{1}{\sigma}}}
\end{array}\right\rangle$$

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

$$\text{I-GIFEHG}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\langle u_2,\Re_2\rangle,\langle u_3,\Re_3\rangle,\ldots,\langle u_n,\Re_n\rangle\right)$$

$$=\left\langle \begin{array}{c} \dfrac{\left(\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}+3\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}\right\}^{\Im_j}+3\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}-\left(\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}+3\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}\right\}^{\Im_j}-\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}}{\left(\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}+3\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}\right\}^{\Im_j}+3\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}+\left(\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}+3\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}\right\}^{\Im_j}-\prod_{j=1}^{n}\left\{\left(1+\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}-\left(1-\xi_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}}, \\ \dfrac{2\left\{\left(\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}\right\}^{\Im_j}-\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}\right\}}{\left(\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}\right\}^{\Im_j}+3\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}+\left(\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}+3\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}\right\}^{\Im_j}-\prod_{j=1}^{n}\left\{\left(2-\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}-\left(\kappa_{\dot{\Re}_{\delta(j)}}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}} \end{array}\right\rangle$$

Definition 11: The Induced generalized intuitionistic fuzzy Einstein hybrid averaging (I-GIFEHA) operator can be defined as follows:

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

$$\text{I-GIFEHA}_{\partial,\Im}\left(\langle u_1,\Re_1\rangle,\langle u_2,\Re_2\rangle,\langle u_3,\Re_3\rangle,\ldots,\langle u_n,\Re_n\rangle\right)$$

$$=\left\langle \begin{array}{l} \dfrac{2\left\{\prod_{j=1}^{n}\left\{\left(2-\xi_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}+3\left(\xi_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}\right\}^{\Im_j}-\prod_{j=1}^{n}\left\{\left(2-\xi_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}-\left(\xi_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}\right\}^{\Im_j}\right\}^{\frac{1}{\sigma}}}{\left(\prod_{j=1}^{n}\left\{\left(2-\xi_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}+3\left(\xi_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}\right\}^{\Im_j}+3\prod_{j=1}^{n}\left\{\left(2-\xi_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}-\left(\xi_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}+\left(\prod_{j=1}^{n}\left\{\left(2-\xi_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}+3\left(\xi_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}\right\}^{\Im_j}-\prod_{j=1}^{n}\left\{\left(2-\xi_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}-\left(\xi_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}}, \\ \dfrac{\left(\prod_{j=1}^{n}\left\{\left(1+\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}+3\left(1-\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}\right\}^{\Im_j}+3\prod_{j=1}^{n}\left\{\left(1+\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}-\left(1-\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}-\left(\prod_{j=1}^{n}\left\{\left(1+\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}+3\left(1-\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}\right\}^{\Im_j}-\prod_{j=1}^{n}\left\{\left(1+\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}-\left(1-\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}}{\left(\prod_{j=1}^{n}\left\{\left(1+\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}+3\left(1-\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}\right\}^{\Im_j}+3\prod_{j=1}^{n}\left\{\left(1+\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}-\left(1-\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}+\left(\prod_{j=1}^{n}\left\{\left(1+\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}+3\left(1-\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}\right\}^{\Im_j}-\prod_{j=1}^{n}\left\{\left(1+\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}-\left(1-\kappa_{\dot{\Re}_{\sigma(j)}}\right)^{\sigma}\right\}^{\Im_j}\right)^{\frac{1}{\sigma}}} \end{array}\right\rangle \quad (35)$$

where $\dot{\Re}_{\delta(j)}$ is the weighted intuitionistic fuzzy value $\dot{\Re}_j\left(\dot{\Re}_j = n\partial_j\Re_j, j=1,2,\ldots,n\right)$ of the intuitionistic fuzzy ordered weighted averaging (IFOWA) pair $\langle u_j,\Re_j\rangle$ having the jth largest u_j in $\langle u_j,\Re_j\rangle$, $\Im=(\Im_1,\Im_2,\ldots,\Im_n)^T$ be the associated vector of I-GIFEHA operator with $\partial_j\in[0,1]$ and $\sum_{j=1}^{n}\partial_j=1$. $\partial=(\partial_1,\partial_2,\ldots,\partial_n)^T$ be the weighted with $\partial_j\in[0,1]$ and $\sum_{j=1}^{n}\partial_j=1$. And σ is a real number greater than zero.

5. Application of the Proposed Methods

In this section, a MAGDM approach has been developed to show the applications and advantages of the new developed methods. For this purpose, we construct and algorithms and also practical example for the selection of new system of information.

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

Algorithms: Let the set of n finite alternatives with $\text{Þ}=\{\text{Þ}_1,\text{Þ}_2,\text{Þ}_3,...\text{Þ}_n\}$, set of m criteria/attributes with $¥=\{¥_1,¥_2,¥_3,...,¥_m\}$ and the set of k experts with $E=\{E_1,E_2,...,E_k\}$.

Let $\Im=(\Im_1,\Im_2,...,\Im_n)^T$ be the associated, $\partial=(\partial_1,\partial_2,...,\partial_n)^T$ be the weighted vector and $\wp=(\wp_1,\wp_2,...,\wp_k)^T$ be weight of decision makers, all have the same conditions that is belongs to the close interval and their sum is equal to 1.

Step 1: Construct decision-making matrices, $E^s=\left[a_{ij}^{(h)}\right]_{m\times n}\ (h=1,2,...,k)$ for decision.

Step 2: If the criteria have two types such as, benefit criteria and cost criteria, then the interval-valued intuitionistic fuzzy decision-making matrice $E^s=\left[a_{ij}^{(h)}\right]_{m\times n}\ (h=1,2,...,k)$ can be converted into the normalized Pythagorean fuzzy decision-making matrices, $R^h=\left[r_{ij}^{(h)}\right]_{m\times n}\ (h=1,2,...,k)$, where

$$r_{ij}^{(h)}=\begin{cases} a_{ij}^{(h)}, \text{ for benefit criteria } \mathrm{C}_j \\ \bar{a}_{ij}^{(h)}, \text{ for cost criteria } \mathrm{C}_j, \end{cases}\begin{pmatrix} j=1,2,...,n \\ i=1,2,...,m \end{pmatrix}$$

and $\bar{a}_{ij}^{(s)}$ is the complement of α_{ij}^s. If all the criteria have the same type, then there is no need of normalization.

Step 3: Utilize the proposed aggregation operator to aggregate all the individual normalized intuitionistic fuzzy decision-making matrices, $R^h=\left[r_{ij}^{(h)}\right]_{m\times n}\ (h=1,2,...,k)$ into a single intuitionistic fuzzy decision-making matrix, $R=\left[r_{ij}\right]_{m\times n}$.

Step 4: Calculate $\dot{\Re}_{ij}=n\partial_j\Re_{ij}$.

Step 5: Again utilize the proposed operators to derive the overall preference values, and then calculate the scores of all preference values

Step 6: That alternative/choice will be consider having the high score function.

6. Illustrative Example

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

Supposing in Abdul Wali Khan University Mardan, the department of computer science wishes to introduce a new system for information in the university. In the first collection, only four systems of information (alternatives) $\text{Þ}_m\left(m=1,2,3,4\right)$ have been considered for further process. For the selection of best option three experts from a group to act as decision makers, and their weight is $\wp=(0.30,0.30,0.40)^T$. For the selecting of most suitable system the experts considered only four criteria, whose weighted vector is $\Im=\left(0.10,0.20,0.30,0.40\right)^T$. $¥_1$: Prices of instrument, $¥_2$: Funding of the university, $¥_3$: Effort to transform from current systems, $¥_4$: Outsourcing software developer reliability, where $¥_1$, $¥_3$ are cost criteria and $¥_2$, $¥_4$ benefit criteria.

Step 1: Construct decision-making matrices

Table 1. Decision Matrix E1.

	$¥_1$	$¥_2$	$¥_3$	$¥_4$
Þ_1	$\langle 0.80,(0.50,0.40)\rangle$	$\langle 0.70,(0.50,0.40)\rangle$	$\langle 0.60,(0.60,0.40)\rangle$	$\langle 0.50,(0.60,0.40)\rangle$
Þ_2	$\langle 0.90,(0.60,0.30)\rangle$	$\langle 0.80,(0.40,0.50)\rangle$	$\langle 0.60,(0.60,0.40)\rangle$	$\langle 0.60,(0.60,0.30)\rangle$
Þ_3	$\langle 0.70,(0.50,0.40)\rangle$	$\langle 0.60,(0.50,0.30)\rangle$	$\langle 0.50,(0.50,0.30)\rangle$	$\langle 0.40,(0.40,0.30)\rangle$
Þ_4	$\langle 0.60,(0.60,0.30)\rangle$	$0.50,\langle(0.50,0.40)\rangle$	$\langle 0.40,(0.50,0.40)\rangle$	$\langle 0.30,(0.50,0.40)\rangle$

Table 2. Decision Matrix E2.

	$¥_1$	$¥_2$	$¥_3$	$¥_4$
Þ_1	$\langle 0.70,(0.60,0.30)\rangle$	$\langle 0.60,(0.50,0.40)\rangle$	$\langle 0.50,(0.60,0.30)\rangle$	$\langle 0.40,(0.60,0.40)\rangle$
Þ_2	$\langle 0.80,(0.50,0.30)\rangle$	$\langle 0.70,(0.60,0.40)\rangle$	$\langle 0.60,(0.40,0.50)\rangle$	$\langle 0.50,(0.50,0.30)\rangle$
Þ_3	$\langle 0.60,(0.60,0.30)\rangle$	$\langle 0.50,(0.60,0.30)\rangle$	$\langle 0.40,(0.60,0.40)\rangle$	$\langle 0.30,(0.50,0.30)\rangle$
Þ_4	$\langle 0.50,(0.40,0.50)\rangle$	$\langle 0.40,(0.60,0.20)\rangle$	$\langle 0.30,(0.40,0.50)\rangle$	$\langle 0.20,(0.40,0.50)\rangle$

Table 3. Decision Matrix of E3.

	$¥_1$	$¥_2$	$¥_3$	$¥_4$

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

p_1	$\langle 0.60,(0.50,0.40)\rangle$	$\langle 0.50,(0.50,0.40)\rangle$	$\langle 0.40,(0.50,0.40)\rangle$	$\langle 0.30,(0.40,0.40)\rangle$
p_2	$\langle 0.70,(0.50,0.30)\rangle$	$\langle 0.60,(0.40,0.30)\rangle$	$\langle 0.50,(0.50,0.30)\rangle$	$\langle 0.40,(0.60,0.30)\rangle$
p_3	$\langle 0.50,(0.60,0.30)\rangle$	$\langle 0.40,(0.60,0.30)\rangle$	$\langle 0.30,(0.50,0.40)\rangle$	$\langle 0.20,(0.60,0.3)0\rangle$
p_4	$\langle 0.40,(0.60,0.20)\rangle$	$\langle 0.30,(0.50,0.40)\rangle$	$\langle 0.20,(0.70,0.10)\rangle$	$\langle 0.10,(0.60,0.30)\rangle$

Step 2: Construct normalized decision-making matrices

Table 4. Normalized Decision Matrix R1.

	¥$_1$	¥$_2$	¥$_3$	¥$_4$
p_1	$\langle 0.80,(0.40,0.50)\rangle$	$\langle 0.70,(0.50,0.40)\rangle$	$\langle 0.60,(0.40,0.60)\rangle$	$\langle 0.50,(0.60,0.40)\rangle$
p_2	$\langle 0.90,(0.30,0.60)\rangle$	$\langle 0.80,(0.40,0.50)\rangle$	$\langle 0.70,(0.50,0.40)\rangle$	$\langle 0.60,(0.60,0.30)\rangle$
p_3	$\langle 0.70,(0.40,0.50)\rangle$	$\langle 0.60,(0.50,0.30)\rangle$	$\langle 0.50,(0.30,0.50)\rangle$	$\langle 0.40,(0.40,0.30)\rangle$
p_4	$\langle 0.60,(0.30,0.60)\rangle$	$0.50,\langle(0.50,0.40)\rangle$	$\langle 0.40,(0.40,0.50)\rangle$	$\langle 0.30,(0.50,0.40)\rangle$

Table 5. Normalized Decision Matrix R2.

	¥$_1$	¥$_2$	¥$_3$	¥$_4$
p_1	$\langle 0.70,(0.30,0.60)\rangle$	$\langle 0.60,(0.50,0.40)\rangle$	$\langle 0.50,(0.30,0.60)\rangle$	$\langle 0.40,(0.60,0.40)\rangle$
p_2	$\langle 0.80,(0.30,0.50)\rangle$	$\langle 0.70,(0.60,0.40)\rangle$	$\langle 0.60,(0.50,0.40)\rangle$	$\langle 0.50,(0.50,0.30)\rangle$
p_3	$\langle 0.60,(0.30,0.50)\rangle$	$\langle 0.50,(0.60,0.30)\rangle$	$\langle 0.40,(0.40,0.60)\rangle$	$\langle 0.30,(0.50,0.30)\rangle$
p_4	$\langle 0.50,(0.50,0.40)\rangle$	$\langle 0.40,(0.60,0.20)\rangle$	$\langle 0.30,(0.50,0.40)\rangle$	$\langle 0.20,(0.40,0.50)\rangle$

Table 6. Normalized Decision Matrix R3.

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

	¥1	¥2	¥3	¥4
p_1	$\langle 0.60,(0.40,0.50)\rangle$	$\langle 0.50,(0.50,0.40)\rangle$	$\langle 0.40,(0.40,0.50)\rangle$	$\langle 0.30,(0.40,0.40)\rangle$
p_2	$\langle 0.70,(0.30,0.50)\rangle$	$\langle 0.60,(0.40,0.30)\rangle$	$\langle 0.40,(0.30,0.50)\rangle$	$\langle 0.40,(0.60,0.30)\rangle$
p_3	$\langle 0.50,(0.30,0.60)\rangle$	$\langle 0.40,(0.60,0.30)\rangle$	$\langle 0.30,(0.40,0.50)\rangle$	$\langle 0.20,(0.60,0.3)\rangle$
p_4	$\langle 0.40,(0.20,0.60)\rangle$	$\langle 0.30,(0.50,0.40)\rangle$	$\langle 0.20,(0.10,0.70)\rangle$	$\langle 0.10,(0.60,0.30)\rangle$

Step 3: Utilize the I-IFEWA operator, where $\wp=(0.30,0.30,0.40)^T$. Then we have

Table 7. Collective Decision Matrix R.

	¥1	¥2	¥3	¥4
p_1	$(0.3716,0.5288)$	$(0.5000,0.4000)$	$(0.3716,0.5581)$	$(0.5262,0.4000)$
p_2	$(0.3000,0.5288)$	$(0.4657,0.3833)$	$(0.4246,0.4380)$	$(0.5416,0.3000)$
p_3	$(0.3308,0.5688)$	$(0.5717,0.3000)$	$(0.3912,0.5288)$	$(0.5512,0.3000)$
p_4	$(0.3263,0.5343)$	$(0.5316,0.3277)$	$(0.2861,0.5753)$	$(0.5717,0.4000)$

Step 4: Calculate $\dot{\Re}_{ij}=n\partial_j\Re_{ij}$, where $\partial=(0.10,0.20,0.30,0.40)^T$. Then we have

$\dot{\Re}_{11}=(0.154,0.798),\dot{\Re}_{12}=(0.415,0.496),\dot{\Re}_{13}=(0.436,0.466),\dot{\Re}_{14}=(0.733,0.196),\dot{\Re}_{21}=(0.123,0.798)$
$\dot{\Re}_{22}=(0.383,0.480),\dot{\Re}_{23}=(0.450,0.357),\dot{\Re}_{24}=(0.748,0.117),\dot{\Re}_{31}=(0.136,0.817),\dot{\Re}_{32}=(0.477,0.399)$
$\dot{\Re}_{33}=(0.458,0.453),\dot{\Re}_{34}=(0.758,0.117),\dot{\Re}_{41}=(0.134,0.800),\dot{\Re}_{42}=(0.441,0.427),\dot{\Re}_{43}=(0.339,0.503)$

$\dot{\Re}_{44}=(0.777,0.196)$. Now we calculate the score functions:

$S(\dot{\Re}_{11})=0.154-0.798=-644, S(\dot{\Re}_{12})=0.415-0.496=-0.081, S(\dot{\Re}_{13})=0.436-0.466=-0.030, S(\dot{\Re}_{14})=0.733-0.196=0.537$
$S(\dot{\Re}_{21})=0.123-0.798=-0.675, S(\dot{\Re}_{22})=0.383-0.480=-0.097, S(\dot{\Re}_{23})=0.450-0.357=0.093, S(\dot{\Re}_{24})=0.748-0.117=0.631$

$S(\dot{\Re}_{31})=0.136-0.817=-0.681, S(\dot{\Re}_{32})=0.477-0.399=0.078, S(\dot{\Re}_{33})=0.458-0.453=0.005, S(\dot{\Re}_{34})=0.758-0.117=0.641$
$S(\dot{\Re}_{41})=0.134-0.800=-0.666, S(\dot{\Re}_{42})=0.441-0.427=0.014, S(\dot{\Re}_{43})=0.339-0.503=-0.016, S(\dot{\Re}_{44})=0.777-0.196=0.581$

Table 8. Hybrid Decision Matrix R.

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

	¥1	¥2	¥3	¥4
p_1	$(0.733, 0.196)$	$(0.436, 0.466)$	$(0.415, 0.496)$	$(0.154, 0.798)$
p_2	$(0.748, 0.117)$	$(0.450, 0.357)$	$(0.383, 0.480)$	$(0.123, 0.798)$
p_3	$(0.758, 0.117)$	$(0.477, 0.399)$	$(0.458, 0.453)$	$(0.136, 0.817)$
p_4	$(0.777, 0.196)$	$(0.441, 0.427)$	$(0.339, 0.503)$	$(0.134, 0.800)$

Step 5: Utilize the I-IFEHA operator, where $\Im = (0.10, 0.20, 0.30, 0.40)^T$. Then we have

$$r_1 = (0.3637, 0.5520), r_2 = (0.3510, 0.5024), r_3 = (0.3858, 0.5584), r_4 = (0.3042, 0.5494)$$

$$S(r_1) = 0.363 - 0.552 = -0.189, S(r_2) = 0.351 - 0.502 = -0.151, S(r_3) = 0.385 - 0.558 = -0.173, S(r_4) = 0.304 - 0.549 = -0.245$$

Step 6: Thus the best option is p_2 .

Table 9. Comparative analysis with some existing methods.

σ	Su et.al. [37]		Rahman et.al. [39]		Proposed Methods	
	Methods	Ranking	Methods	Ranking	Methods	Ranking
1	I-IFOWA	(2134)	IFEHA	(2134)	I-IFEHA	(2134)
	I-IFOWG	(2134)	IFEHG	(2134)	I-IFEHG	(2134)
	I-IFHA	(2134)	GIFEHA	(2134)	I-GIFEHA	(2134)
	I-IFHG	(2134)	GIFEHG	(2134)	I-GIFEHG	(2134)
2	I-IFOWA	(2134)	IFEHA	(2134)	I-IFEHA	(2134)
	I-IFOWG	(2134)	IFEHG	(2134)	I-IFEHG	(2134)
	I-IFHA	(2134)	GIFEHA	(2134)	I-GIFEHA	(2134)
	I-IFHG	(2134)	GIFEHG	(2134)	I-GIFEHG	(2134)
3	I-IFOWA	(2134)	IFEHA	(2134)	I-IFEHA	(2134)
	I-IFOWG	(2134)	IFEHG	(2134)	I-IFEHG	(2134)
	I-IFHA	(2134)	GIFEHA	(1234)	I-GIFEHA	(2134)
	I-IFHG	(2134)	GIFEHG	(1234)	I-GIFEH	(2134)
4	I-IFOWA	(2134)	IFEHA	(2134)	I-IFEHA	(2134)
	I-IFOWG	(2134)	IFEHG	(2134)	I-IFEHG	(2134)
	I-IFHA	(2134)	GIFEHA	(2134)	I-GIFEHA	(2134)
	I-IFHG	(2134)	GIFEHG	(2134)	I-GIFEHG	(2134)

7. Conclusion

Generalized Einstein operators syndicate Einstein operators with some generalized operators using IFNs. Hence Einstein operational laws play an imperative, significant and important role in our daily problems in the case of best selection. Therefore, here we have presented and explore some new types methods and operators using on IFNs base

on Einstein operations, such as I-IFEHA operator, I-IFEHG operator, I-GIFEHA operator, I-GIFEHG operator. Additionally, the new methods have been utilized and exploited with decision-making. We have discussed the applicability in a decision-making concerning strategic choice of the best information system and also construct examples and the operational processes were illustrated in detail to show the efficacy and value of the developed new methods. Some of their wanted and structure properties, such as monotonicity, boundedness idempotency, commutativity were developed and some new results were developed. Finally, an illustrative example is given to show the steps of decision process of the proposed operators and methods.

In further research, it is necessary to give the applications of these operators to the other domains such as, Confidence levels, Hamacher operators, Power operators, Symmetric operator, Logarithmic operators, Dombi operators, Linguistic terms, trigonometric operation, ranking method for normal intuitionistic sets.

References

1. Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353.

2. Atanassov, K.T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets Syst*. 20(1), 87–96.

3. Atanassov, K. T. (1989). More on intuitionistic fuzzy sets. *Fuzzy sets and systems*, *33*(1), 37-45.

4. Garg, H. (2020). Exponential operational laws and new aggregation operators for intuitionistic multiplicative set in multiple-attribute group decision making process. *Information Sciences*.

5. Garg, H., Arora, R. (2019). Generalized intuitionistic fuzzy soft power aggregation operator based on t-norm and their application in multicriteria decision-making. *International Journal of Intelligent Systems*, *34*(2), 215-246.

6. Garg, H., Kumar, K. (2020). A novel possibility measure to interval-valued intuitionistic fuzzy set using connection number of set pair analysis and its applications. *Neural Computing and Applications*, *32*(8), 3337-3348.

7. Xu, Z. S. (2007). Models for multiple attribute decision making with intuitionistic fuzzy information. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *15*(03), 285-297.

8. Yu, D., Wu, Y., Lu, T. (2012). Interval-valued intuitionistic fuzzy prioritized operators and their application in group decision making. *Knowledge-Based Systems*, *30*, 57-66.

9. Bustince, H., Burillo, P. (1995). Correlation of interval-valued intuitionistic fuzzy sets. *Fuzzy sets and systems*, *74*(2), 237-244.

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

10. Xu, Z., Xia, M. (2011). Induced generalized intuitionistic fuzzy operators. *Knowledge-Based Systems*, *24*(2), 197-209.

11. Tan, C. (2011). Generalized intuitionistic fuzzy geometric aggregation operator and its application to multi-criteria group decision making. *Soft Computing*, *15*(5), 867-876.

12. Xia, M., Xu, Z. (2010). Some new similarity measures for intuitionistic fuzzy values and their application in group decision making. *Journal of Systems Science and Systems Engineering*, *19*(4), 430-452.

13. Li, D. F. (2011). The GOWA operator based approach to multiattribute decision making using intuitionistic fuzzy sets. *Mathematical and Computer Modelling*, *53*(5-6), 1182-1196.

14. Wei, G. W. (2008). Maximizing deviation method for multiple attribute decision making in intuitionistic fuzzy setting. *Knowledge-Based Systems*, *21*(8), 833-836.

15. Wei, G. W. (2010). GRA method for multiple attribute decision making with incomplete weight information in intuitionistic fuzzy setting. *Knowledge-Based Systems*, *23*(3), 243-247.

16. Wei, G. W. (2011). Gray relational analysis method for intuitionistic fuzzy multiple attribute decision making. *Expert systems with Applications*, *38*(9), 11671-11677.

17. Wei, G., Zhao, X. (2011). Minimum deviation models for multiple attribute decision making in intuitionistic fuzzy setting. *International Journal of Computational Intelligence Systems*, *4*(2), 174-183.

18. Wei, G. W., Wang, H. J., Lin, R. (2011). Application of correlation coefficient to interval-valued intuitionistic fuzzy multiple attribute decision-making with incomplete weight information. *Knowledge and Information Systems*, *26*(2), 337-349.

19. Grzegorzewski, P. (2004). Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdorff metric. *Fuzzy sets and systems*, *148*(2), 319-328.

20. Yager, R. R. (2009). Some aspects of intuitionistic fuzzy sets. *Fuzzy Optimization and Decision Making*, *8*(1), 67-90.

21. Ye, J. (2010). Fuzzy decision-making method based on the weighted correlation coefficient under intuitionistic fuzzy environment. *European Journal of Operational Research*, *205*(1), 202-204.

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

22. Ye, J. (2011). Fuzzy cross entropy of interval-valued intuitionistic fuzzy sets and its optimal decision-making method based on the weights of alternatives. *Expert Systems with Applications*, *38*(5), 6179-6183.

23. Garg, H. (2016). Generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein t-norm and t-conorm and their application to decision making. *Computers & Industrial Engineering*, *101*, 53-69.

24. Garg, H. (2019). Intuitionistic fuzzy Hamacher aggregation operators with entropy weight and their applications to multicriteria decision-making problems. *Iranian Journal of Science and Technology - Transactions of Electrical Engineering*, 597-613.

25. Garg, H., Kumar, K. (2019). Power geometric aggregation operators based on connection number of set pair analysis under intuitionistic fuzzy environment, *Arabian Journal for Science and Engineering*.

26. Garg, H., Kumar, K. (2018). A novel correlation coefficient of intuitionistic fuzzy sets based on the connection number of set pair analysis and its application. *Scientia Iranica. Transaction E, Industrial Engineering*, *25*(4), 2373-2388.

27. Garg, H., Rani, D. (2019). Generalized Geometric Aggregation Operators Based on T-Norm Operations for Complex Intuitionistic Fuzzy Sets and Their Application to Decision-making. *Cognitive Computation*, 1-20.

28. Xu, Z. (2007). Intuitionistic fuzzy aggregation operators. *IEEE Transactions on fuzzy systems*, *15*(6), 1179-1187.

29. Xu, Z., Yager, R. R. (2006). Some geometric aggregation operators based on intuitionistic fuzzy sets. *International journal of general systems*, *35*(4), 417-433.

30. Xu, Z. (2011). Approaches to multiple attribute group decision making based on intuitionistic fuzzy power aggregation operators. *Knowledge-Based Systems*, *24*(6), 749-760.

31. Wang, W., Liu, X. (2011). Intuitionistic fuzzy geometric aggregation operators based on Einstein operations. *International Journal of Intelligent Systems*, *26*(11), 1049-1075.

32. Wang, W., Liu, X. (2012). Intuitionistic fuzzy information aggregation using Einstein operations. *IEEE Transactions on Fuzzy Systems*, *20*(5), 923-938.

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran

33. Zhao, X., Wei, G. (2013). Some intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute decision making. *Knowledge-Based Systems*, *37*, 472-479.

34. Zhao, H., Xu, Z., Ni, M., Liu, S. (2010). Generalized aggregation operators for intuitionistic fuzzy sets. *International journal of intelligent systems*, *25*(1), 1-30.

35. Wei, G. (2010). Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making. *Applied soft computing*, *10*(2), 423-431.

36. Yu, X., Xu, Z. (2013). Prioritized intuitionistic fuzzy aggregation operators. *Information Fusion*, *14*(1), 108-116.

37. Su, Z. X., Xia, G. P., Chen, M. Y. (2011). Some induced intuitionistic fuzzy aggregation operators applied to multi-attribute group decision making. *International Journal of General Systems*, *40*(8), 805-835.

38. Xu, Y., Li, Y., Wang, H. (2013). The induced intuitionistic fuzzy Einstein aggregation and its application in group decision-making. *Journal of Industrial and Production Engineering*, *30*(1), 2-14.

39. Rahman, K., Abdullah, S., Jamil, M., Khan, M. Y. (2018). Some generalized intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute group decision making. *International Journal of Fuzzy Systems*, *20*(5), 1567-1575.

40. Rahman, K., Ayub, S., Abdullah, S. (2020). Generalized intuitionistic fuzzy aggregation operators based on confidence levels for group decision making. *Granular Computing*, 1-20.

41. Jamil, M., Rahman, K., Abdullah, S., Khan, M. Y. (2020). The induced generalized interval-valued intuitionistic fuzzy einstein hybrid geometric aggregation operator and their application to group decision-making. *Journal of Intelligent & Fuzzy Systems*, (Preprint), 1-16.

Published by Firouzabad Institute of Higher Education, Firouzabad, Fars, Iran