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# Picture fuzzy labelling graphs with an application

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**Abstract** The main objective of this paper is to introduce the idea of picture fuzzy labelling of graphs and the concepts of strong arc, partial cut node, bridge of picture fuzzy labelling graphs, picture fuzzy labelling tree and cycle along with their properties and results. In addition, an application of the picture fuzzy graph labelling model for the human circulatory system has been discussed.

**Keywords** Picture fuzzy graph labelling; Strength of connectedness; Picture fuzzy labelling tree; Picture fuzzy labelling cycle; Fuzzy labelling graphs

# PICTURE FUZZY LABELLING GRAPHS WITH AN APPLICATION

**ABSTRACT:** The main objective of this paper is to introduce the idea of picture fuzzy labelling of graphs and the concepts of strong arc, partial cut node, bridge of picture fuzzy labelling graphs, picture fuzzy labelling tree and cycle along with their properties and results. In addition, an application of the picture fuzzy graph labelling model for the human circulatory system has been discussed.

**KEYWORDS:** Picture fuzzy graph labelling, strength of connectedness, picture fuzzy labelling tree, picture fuzzy labelling cycle, fuzzy labelling graphs.

**INTRODUCTION:**

Fuzzy set theory was published by Zadeh [1] in 1965. In 1986, Atanassov [2] established the view of intuitionistic fuzzy set. A set which has the elements with degree of membership is a fuzzy set, whereas the intuitionistic fuzzy set has the degree of membership and non-membership. Intuitionistic fuzzy has many advantages in handling vagueness and uncertainty when compared with the fuzzy set. Smarandache [3] coined the idea of Neutrosophic Set (NS) which is a brand-new dimension to the sets and it was applied in many fields especially in the field of decision making [4-9]. The modified form of fuzzy set and intuitionistic set, the picture fuzzy set was coined by Cuong and Kreinovich [10-11]. Picture fuzzy set (PFS) has three memberships namely, positive, neutral and negative for each element. Again, Coung et al. [12] investigated the fundamental fuzzy operators namely, negations, conjunctions, disjunctions and implications on PFS. Application of fuzzy is used in many fields along with chemical industry, robotics, industrial automation, defence, air conditioners, electronics, power engineering, control image processing, washing machines, structures engineering and facial pattern recognition.

The graph theory concept was presented by Euler in 1736. A graph G is composed of fixed sets of vertices $V(G)$ and edges $E(G)$. In a simple graph, two of the vertices in G are related if there exists an edge $(v_i, v_j)\ in\ E(G)$ connecting the vertices $v_i\ and\ v_j$ in $V(G)$. The idea of graph labelling was introduced by Rosa [13]. The practical troubles are represented via graphs and are applied in many fields which includes Computer science, Electrical Engineering, Mathematics, Physics, Chemistry, Linguistics, Computer Network, Social Sciences, Biology, etc.

Fuzzy graphs from fuzzy relations become introduced with the aid of Kauffmann [14]. It turned to Rosenfeld [15] who advanced fuzzy graphs in 1975. The labelling of fuzzy graphs was done by A.N. Gani et.al [16] and they additionally discussed the properties of FLG. There are many types of fuzzy labelling such as cordial, magic and graceful etc. Fuzzy magic labelling has been applied for fuzzy graphs, intuitionistic fuzzy graphs [17], neutrosophic path and star graphs [18]. Fuzzy graphs have been advanced with the

introduction of intuitionistic fuzzy graphs, neutrosophic graphs, hesitant and so on. Picture fuzzy graph [19] came to be advanced from the picture fuzzy relation.

The crux of this work is to introduce the picture fuzzy labelling of graphs and investigate some of the properties of picture fuzzy labelling. Also we will discuss the application of picture fuzzy labelling to circulatory system model.

## PRELIMINARIES:

**Definition 2.1:** Let $X$ be a fixed set with elements $a$. Then a fuzzy set $A$ in $X$ is defined as $A = \{(a, \mu_A(a)) | \, a \in X)\}$ where $\mu_A(a) \in [0,1]$ is called the membership function for the fuzzy set $A$.

**Definition 2.2:** A fuzzy graph $G = (V, \sigma, \mu)$ with $\sigma: V \to [0,1]$, $\mu: V \times V \to [0,1]$such that for all $a, b \; in \; V, \mu(a,b) \leq \sigma(a) \wedge \sigma(b)$. A fuzzy graph G is called a fuzzy labelling graph if $\mu(a,b) < \sigma(a) \wedge \sigma(b)$.

**Definition 2.3:** An intuitionistic fuzzy set $D$, in a non-void set $X$ is defined as $D = \{(a, \mu_D(a), \gamma_D(a)) \mid a \in X)\}$ where $\mu_D: X \to [0,1]$ and $\gamma_D: X \to [0,1]$ denote the degree of membership and non-membership respectively, and $0 \leq \mu_D(a) + \gamma_D(a) \leq 1$.

**Definition 2.4:** A picture fuzzy set [10] L in X is defined by $L = \{(a, \mu_L(a), \eta_L(a), \gamma_L(a)) \mid a \in X)\}$, where $\mu_L(a)$, $\eta_L(a)$, $\gamma_L(a) \in [0,1]$ denote the positive, neutral and negative membership degree of the element $a$ in L and $\mu_L(a)$, $\eta_L(a), \gamma_L(a)$ follow the condition that $0 \leq \mu_L(a) + \eta_L(a) + \gamma_L(a) \leq 1$.

**Definition 2.5:** $G = (V, \sigma, \mu)$ with $V = \{a_1, a_2, \dots, a_n\}$ (where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$) and $\sigma_1: V \to [0,1], \sigma_2: V \to [0,1], \sigma_3: V \to [0,1]$ and $\mu_1: V \times V \to [0,1], \mu_2: V \times V \to [0,1]$ and $\mu_3: V \times V \to [0,1]$ representing the positive, neutral and negative membership functions of vertices and edges respectively satisfying the condition

$$\mu_1(a_i, a_j) \leq \sigma_1(a_i) \wedge \sigma_1(a_j),$$
$$\mu_2(a_i, a_j) \leq \sigma_2(a_i) \wedge \sigma_2(a_j),$$
$$\mu_3(a_i, a_j) \leq \sigma_3(a_i) \vee \sigma_3(a_j).$$

where $0 \leq \mu_1(a_i, a_j) + \mu_2(a_i, a_j) + \mu_3(a_i, a_j) \leq 1$ for every $a_i, a_j \in V(i, j = 1, 2, \dots, n)$ is said to be a picture fuzzy graph [4] (PFG).

## PICTURE FUZZY LABELLING GRAPHS:

**Definition 3.1:**

A picture fuzzy graph $G = (V, \sigma, \mu)$ is said to be a picture fuzzy labelling graph if $\sigma_1 : V \to [0,1]$, $\sigma_2 : V \to [0,1]$, $\sigma_3 : V \to [0,1]$ and $\mu_1 : V \text{ x } V \to [0,1]$, $\mu_2 : V \text{ x } V \to [0,1]$, $\mu_3 : V \text{ x } V \to [0,1]$ are one to one such that the positive, neutral and negative membership functions of the edges and vertices where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$ are distinct such that

$$\mu_1(a_i, a_{i+1}) < \sigma_1(a_i) \wedge \sigma_1(a_{i+1})$$
$$\mu_2(a_i, a_{i+1}) < \sigma_2(a_i) \wedge \sigma_2(a_{i+1})$$
$$\mu_3(a_i, a_{i+1}) < \sigma_3(a_i) \vee \sigma_3(a_{i+1})$$

$0 \leq \mu_1(a_i, a_{i+1}) + \mu_2(a_i, a_{i+1}) + \mu_3(a_i, a_{i+1}) \leq 1 \; for\ all\ edges\ (a_i, a_{i+1}) \in V \text{ x } V.$

**Note:**

In the following theorems and results,

- Picture fuzzy labelling graph, picture fuzzy labelling and picture fuzzy will be denoted as PFLG, PFL and PF respectively.
- Positive, Neutral and Negative membership function will be denoted respectively as PM, NM and NEM.

**Example 3.2:**

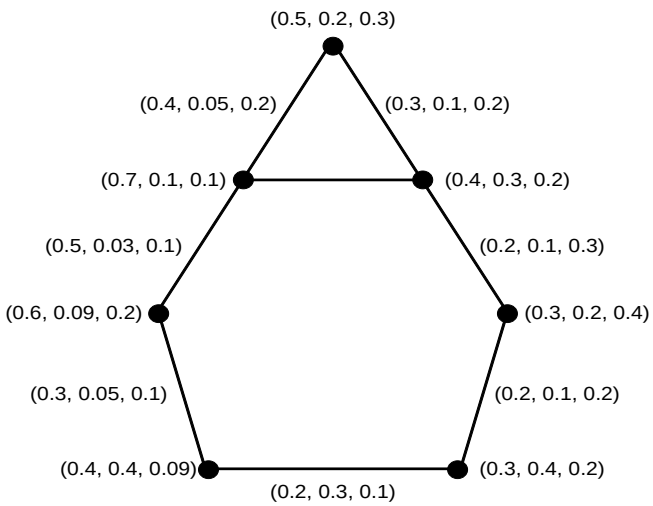


**Figure 1** Picture Fuzzy Labelling Graph

**Definition 3.3:**

Let $G = (V, \sigma, \mu)$ be a PFLG. $H = (V, \alpha, \beta)$ with $\alpha_1 : V \to [0,1]$, $\alpha_2 : V \to [0,1]$, $\alpha_3 : V \to [0,1]$ and $\beta_1 : V \text{ x } V \to [0,1]$, $\beta_2 : V \text{ x } V \to [0,1]$, $\beta_3 : V \text{ x } V \to [0,1]$ as PM, NM, NEM of vertices and edges respectively where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ is said to be a picture fuzzy labelling subgraph of $G$.

If $\alpha_1(a) \leq \sigma_1(a)$, $\alpha_2(a) \leq \sigma_2(a)$ & $\alpha_3(a) \geq \sigma_3(a)$ $\forall\, a \in V$ and

$\beta_1(a,b) \leq \mu_1(a,b)$, $\beta_2(a,b) \leq \mu_2(a,b)$ & $\beta_3(a,b) \geq \mu_3(a,b)$ $\forall\, (a,b) \in V \text{ x } V$

**Theorem 3.4:**

If $H = (V, \alpha, \beta)$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ is a PFL subgraph of $G = (V, \sigma, \mu)$ where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$ then

$$\beta_1^\infty(a,b) \le \mu_1^\infty(a,b),$$
$$\beta_2^\infty(a,b) \le \mu_2^\infty(a,b),$$
$$\beta_3^\infty(a,b) \ge \mu_3^\infty(a,b). \forall\, a,b \in V$$

where $\beta_1^\infty$, $\beta_2^\infty$, $\beta_3^\infty$ and $\mu_1^\infty$, $\mu_2^\infty$, $\mu_3^\infty$ are the strength of the picture fuzzy labelling graph G and picture fuzzy labelling subgraph H of G.

***Proof:***

Let $G = (V, \sigma, \mu)$ be any PFLG and $H = (V, \alpha, \beta)$ be its PFL subgraph.
Let $(a, b)$ be any PF path in G and its strength be $\mu_1^\infty(a,b)$, $\mu_2^\infty(a,b)$, $\mu_3^\infty(a,b)$.
Since H is a PFL subgraph of G,

$$\alpha_1(a) \le \sigma_1(a), \beta_1(a,b) \le \mu_1(a,b),$$
$$\alpha_2(a) \le \sigma_2(a), \beta_2(a,b) \le \mu_2(a,b),$$
$$\alpha_3(a) \ge \sigma_3(a), \beta_3(a,b) \ge \mu_3(a,b).$$
$$\forall\, a,b \in V \text{ and } (a,b) \in E \subseteq V \times V$$

which implies that,

$$\beta_1^\infty(a,b) \le \mu_1^\infty(a,b),$$
$$\beta_2^\infty(a,b) \le \mu_2^\infty(a,b),$$
$$\beta_3^\infty(a,b) \ge \mu_3^\infty(a,b). \forall\, a,b \in V$$

***Theorem 3.5:***

The union of two PFLG is also a PFLG.

***Proof:***

Let $G'$=$(V', \sigma', \mu')$ and $G''$=$(V'', \sigma'', \mu'')$ be any two PFLGs such that the PM, NM and NEM values of the edges of $G'$ & $G''$ are distinct. Let $G = (V, \sigma, \mu)$ with $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$ be the union of $G'$ & $G''$.

Now, we prove that G is a PFLG.

$$\sigma_1(a) = \begin{cases} \sigma_1'(a) & if\ \ a \in V' - V'' \\ \sigma_1''(a) & if\ \ a \in V'' - V' \\ \sigma_1'(a) \vee \sigma_1''(a) & if\ \ a \in V' \cap V'' \end{cases}$$
$$\sigma_2(a) = \begin{cases} \sigma_2'(a) & if\ \ a \in V' - V'' \\ \sigma_2''(a) & if\ \ a \in V'' - V' \\ \sigma_2'(a) \vee \sigma_2''(a) & if\ \ a \in V' \cap V'' \end{cases}$$
$$\sigma_3(a) = \begin{cases} \sigma_3'(a) & if\ \ a \in V' - V'' \\ \sigma_3''(a) & if\ \ a \in V'' - V' \\ \sigma_3'(a) \wedge \sigma_3''(a) & if\ \ a \in V' \cap V'' \end{cases}$$

And

$$\mu_1(a,b) = \begin{cases} \mu_1'(a,b) & if\ \ (a,b) \in E' - E'' \\ \mu_1''(a,b) & if\ \ (a,b) \in E'' - E' \\ \mu_1'(a,b) \vee \mu_1''(a,b) & if\ \ (a,b) \in E' \cap E'' \end{cases}$$

$$\mu_2(a,b) = \begin{cases} \mu_2'(a,b) & if\ \ (a,b) \in E' - E'' \\ \mu_2''(a,b) & if\ \ (a,b) \in E'' - E' \\ \mu_2'(a,b) \vee \mu_2''(a,b) & if\ \ (a,b) \in E' \cap E'' \end{cases}$$

$$\mu_3(a,b) = \begin{cases} \mu_3'(a,b) & if\ \ (a,b) \in E' - E'' \\ \mu_3''(a,b) & if\ \ (a,b) \in E'' - E' \\ \mu_3'(a,b) \wedge \mu_3''(a,b) & if\ \ (a,b) \in E' \cap E'' \end{cases}$$

The PM, NM and NEM of the edges and vertices are distinct. And the defined $G = (V, \sigma, \mu)$ satisfies the condition of PFL. Hence union of two PFLG is also a PFLG.

**Definition 3.6:**

Let $G = (V, \sigma, \mu)$ be a PFLG. The strength of the picture fuzzy path P of n edges $e_i$ for $i = 1, \dots n$ is denoted by $PS(P) = (PS_1(P), PS_2(P),\ PS_3(P))$ and is defined by

$$PS_1(P) = \min_{1 \le i \le n} \mu_1(e_i),$$
$$PS_2(P) = \min_{1 \le i \le n} \mu_2(e_i),$$
$$PS_3(P) = \max_{1 \le i \le n} \mu_3(e_i).$$

**Definition 3.7:**

Let $G = (V, \sigma, \mu)$ be a PFLG. The picture fuzzy strength of connectedness of a duo of vertices $a, b \in V$, denoted by $PCONN_G(a,b) = (PCONN_{1G}(a,b),\ PCONN_{2G}(a,b),\ PCONN_{3G}(a,b))$ is defined by
$PCONN_{1G}(a,b) = max\ \{PS_1(P)/P\ is\ a\ a-b\ PF\ path\ in\ G\}$ and $PCONN_{2G}(a,b) = max\ \{PS_2(P)/P\ is\ a\ \ a-b\ PF\ path\ in\ G\}$ and $PCONN_{3G}(a,b) = min\ \{PS_3(P)/P\ is\ a\ \ a-b\ PF\ path\ in\ G\}$. $PCONN_G(a,b) = (0,0,0)$ if the vertices are is defined in G.

**Example 3.8:**

Consider the following PFLG, $G = (V, \sigma, \mu)$.

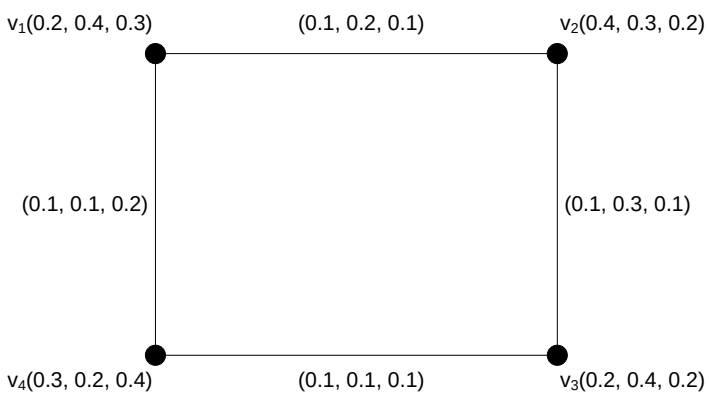


**Figure 2** Picture Fuzzy Strength of Connectedness

$$PCONN_G(v_1, v_2) = (0.1, 0.2, 0.1)$$
$$PCONN_G(v_1, v_3) = (max\ (0.1, 0.1), max\ (0.2, 0.1), \min(0.1, 0.2))$$

$$= (0.1, 0.2, 0.1)$$
$$PCONN_G(v_1, v_4) = (max\ (0.1, 0.1), max\ (0.1, 0.1), \min(0.1, 0.2))$$
$$= (0.1, 0.1, 0.1)$$

**Proposition 3.9:**

Let $G$ be a PFLG and $H$ a PFL subgraph of $G$. Then for every pair of vertices $a, b \in V$, we have

$$PCONN_{1H}(a, b) \leq PCONN_{1G}(a, b),$$
$$PCONN_{2H}(a, b) \leq PCONN_{2G}(a, b),$$
$$PCONN_{3H}(a, b) \geq PCONN_{3G}(a, b).$$

**Definition 3.10:**

A $a - b$ picture fuzzy path in a PFLG is called the strongest $a - b$ PF path if

$$PS_1(P) = PCONN_{1G}(a, b),$$
$$PS_2(P) = PCONN_{2G}(a, b),$$
$$PS_3(P) = PCONN_{3G}(a, b).$$

**Definition 3.11:**

Let $G$ be a PFLG. A node $w$ is called a picture fuzzy partial cut node $(p - cut\ node)$ of $G$ if there exists a pair of nodes $a, b \in V$ such that $a \neq b \neq c$ and

$$PCONN_{1(G-c)}(a, b) < PCONN_{1G}(a, b),$$
$$PCONN_{2(G-c)}(a, b) < PCONN_{2G}(a, b),$$
$$PCONN_{3(G-c)}(a, b) > PCONN_{3G}(a, b).$$

A connected PFLG having no $(p - cut\ node)$ is called a picture fuzzy partial block.

**Example 3.12:**

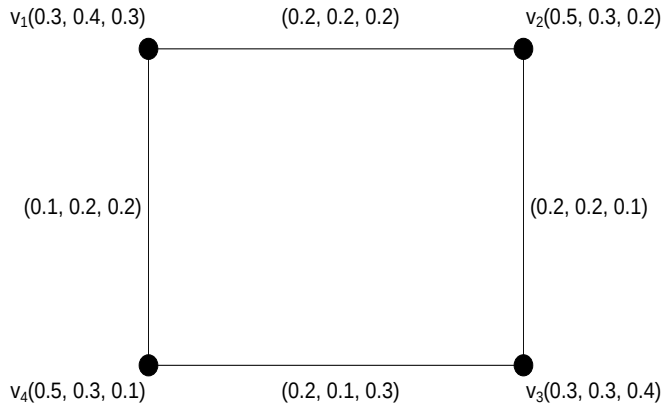


**Figure 3** PFLG with picture fuzzy partial cut node $v_2$

$$PCONN_G(v_1, v_3) = (max(0.2,0.1), max(0.2,0.1), min(0.2,0.3))$$
$$= (0.2, 0.2, 0.2)$$
$$PCONN_{G-v_2}(v_1, v_3) = (0.1, 0.1,0.3)$$

**Definition 3.13:**

Let $G$ be a PFLG. An arc $e = (a, b)$ is called a picture fuzzy partial bridge $(p - bridge)$ if

$$PCONN_{1(G-e)}(a, b) < PCONN_{1G}(a, b),$$
$$PCONN_{2(G-e)}(a, b) < PCONN_{2G}(a, b),$$
$$PCONN_{3(G-e)}(a, b) > PCONN_{3G}(a, b).$$

A $(p - bridge)$ is said to be a picture fuzzy partial bond $(p - bond)$ if

$$PCONN_{1(G-e)}(x, y) < PCONN_{1G}(x, y),$$
$$PCONN_{2(G-e)}(x, y) < PCONN_{2G}(x, y),$$
$$PCONN_{3(G-e)}(x, y) > PCONN_{3G}(x, y).$$

with at least one of $x\ or\ y$ different from $a\ and\ b$ and is said to be a picture fuzzy partial cut bond if both $x\ and\ y$ are different from $a\ and\ b$.

**Example 3.14:**

Here is an example for picture fuzzy partial bridge of PFLG. In this PFLG all the edges except $(v_3, v_4)$ are picture fuzzy partial bridge. Edge $(v_1, v_2)$ is picture fuzzy partial cut bond, since $PCONN_{G-v_1v_2}(v_3, v_4) = (0.01, 0.02,\ 0.5)\ and\ PCONN_G(v_3, v_4) = (0.02, 0.03, 0.4)$.

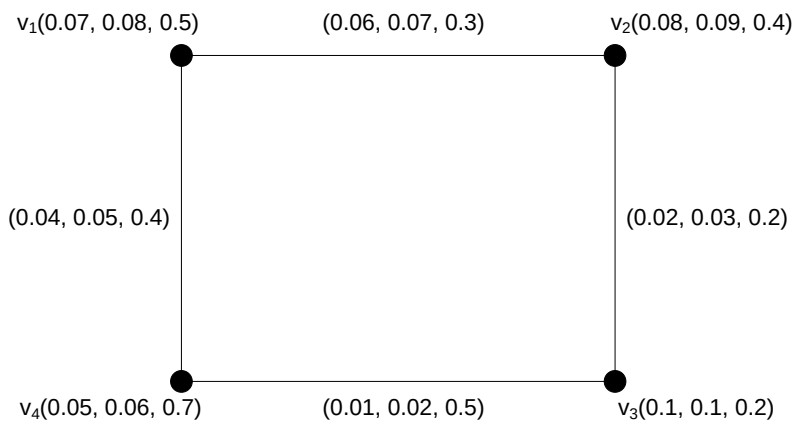


**Figure 4** PFLG with picture fuzzy partial bridge

**Definition 3.15:**

Let $G$ be a PFLG and $C$, a picture fuzzy cycle in G. Then,

(i) $C$ is called a strong picture fuzzy cycle if all picture fuzzy arcs in $C$ are strong.

(ii) A picture fuzzy arc $e = (x, y) \in E$ is called PF $\alpha - strong$ if $PCONN_{1(G-e)}(x, y) < \mu_1(a, b)$,
$PCONN_{2(G-e)}(x, y) < \mu_2(a, b), PCONN_{3(G-e)}(x, y) > \mu_3(a, b)$; a PF $\delta - arc$ if
$PCONN_{1(G-e)}(x, y) > \mu_1(a, b), PCONN_{2(G-e)}(x, y) > \mu_2(a, b), PCONN_{3(G-e)}(x, y) < \mu_3(a, b)$.

(iii) A $a-b$ picture fuzzy path $P$ in $G$ is named a strong $a-b$ picture fuzzy path if all the edges of $P$ are strong.
In particular, if all the PF arcs of $P$ are PF $\alpha - strong,$ then $P$ is called $\alpha - strong$ picture fuzzy path.

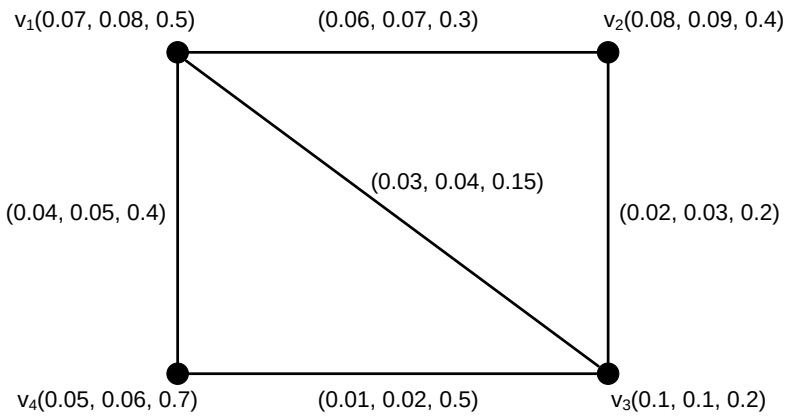


**Figure 5** $v_4v_1v_2$ is an $\alpha-$ strong picture fuzzy path in PFLG

**Definition 3.16:**

A PFLG is named a $\theta$-picture fuzzy labelling graph if for each pair of nodes $a\ and\ b$, either all strong picture fuzzy cycles passing through $a\ and\ b$ have the same strength or there is no strong picture fuzzy cycle passing through $a\ and\ b$.

**Example 3.17:**

Consider the following PFLG, clearly G is a $\theta$-PFLG as G has no strong picture fuzzy cycles.

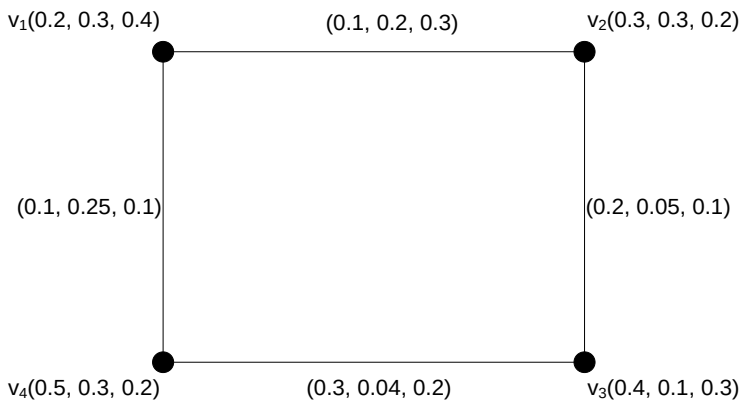


**Figure 6** θ-PFLG

**Theorem 3.18:**

Let $G$ be a connected PFLG and let $a, b$ be 2 nodes in $G$. Then ∃ a strong picture fuzzy path from $a\ to\ b$.

**Proof:**

Suppose that $G = (V, \sigma, \mu)$ is a connected PFLG. Let $a$ and $b$ be any two nodes of $G$. If the picture fuzzy arc $(a, b)$ is strong, then there is nothing to prove. Otherwise, either $(a, b)$ is a PF $\delta - arc$ or there will be a picture fuzzy path having length more than one from $a\ to\ b$.

In the first case, we can find a picture fuzzy path $P$ such that,

$$PS_1(P) > \mu_1(a, b),$$
$$PS_2(P) > \mu_2(a, b),$$
$$PS_3(P) < \mu_3(a, b).$$

In either case, the picture fuzzy path from $a\ to\ b$ is of length more than one. If some picture fuzzy arc on this picture fuzzy path is not strong, replace it by a picture fuzzy path having more strength. Hence there will be a picture fuzzy path from $a\ to\ b$ on which all the picture fuzzy arcs are strong. Hence, ∃ a strong picture fuzzy path from $a\ to\ b$.

**Proposition 3.19:**

Let $G$ be a PFL cycle, then $G$ has exactly one weakest PF arc.

**Proof:**

Let $G = (V, \sigma, \mu)$ be a PFL cycle. Let

$$\mu_1(a, b) = \bigwedge_{i=1}^{n} \mu_1(a_i, b_i)$$
$$\mu_2(a, b) = \bigwedge_{i=1}^{n} \mu_2(a_i, b_i)$$
$$\mu_3(a, b) = \bigvee_{i=1}^{n} \mu_3(a_i, b_i)$$

Since $G$ has PFL, it will have only one PF arc with $\mu(a, b) = \big(\mu_1(a, b), \mu_2(a, b), \mu_3(a, b)\big)$. If we remove $\mu(a, b)$ from $G$, the picture fuzzy strength of connectedness will not be reduced implying that $\mu(a, b)$ is the weakest PF arc. Hence there exists only one weakest picture fuzzy arc in any PFL cycle.

**Proposition 3.20:**

If $G$ is a PFL cycle, then it has $(n - 1)$ picture fuzzy bridges.

**Proof:**

Let $G = (V, \sigma, \mu)$ be a PFLG. By previous proposition, G has only one weakest picture fuzzy arc. We know that weakest picture fuzzy edge is not a picture fuzzy bridge. This implies that the picture fuzzy strength of the connectedness will be reduced whenever any edge except weakest one is removed. Hence every PFLG has $(n - 1)$ picture fuzzy bridges.

**Proposition 3.21:**

If $G$ is a PFLG, then every picture fuzzy bridge is strong and vice versa.

**Proof:**

Let $G = (V, \sigma, \mu)$ be a PFLG with $n$ nodes. $G$ has exactly one weakest picture fuzzy arc and has $n - 1$ picture fuzzy bridges.

Our claim is to prove that $n - 1$ picture fuzzy bridges are strong. Let us choose a picture fuzzy arc $a_i, a_{i+1}$ from $n - 1$ edges. Since $G$ is a picture fuzzy labelling cycle, there are two picture fuzzy paths between the nodes $a_i$ and $a_{i+1}$.

i.e. one picture fuzzy path with $\mu(a_i, a_{i+1}) > 0$ and the other with

$\mu(a_i, a_{i+1}) = (\mu_1(a_i, a_{i+1}), \mu_2(a_i, a_{i+1}), \mu_3(a_i, a_{i+1}))$

$$\mu(a_i, \dots\ a_{i+n}, \dots\ a_{i+1}) = (\mu_1(a_i, \dots\ a_{i+n}, \dots\ a_{i+1}), \mu_2(a_i, \dots\ a_{i+n}, \dots\ a_{i+1}), \mu_3(a_i, \dots\ a_{i+n}, \dots\ a_{i+1})) > 0$$

$\mu^{\infty}(a_i, a_{i+1}) = \mu(a_i, a_{i+1}) = (\mu_1(a_i, a_{i+1}), \mu_2(a_i, a_{i+1}), \mu_3(a_i, a_{i+1}))$

which implies that $(a_i, a_{i+1})$ is a strong picture fuzzy arc. Repeating this argument for all $(n - 1)$ edges, we obtain that every picture fuzzy bridge is strong. The converse of the theorem is obvious.

**Proposition 3.22:**

Let $G$ be a PFLG. Then $G$ has at least one picture fuzzy bridge.

**Proof:**

Let $G = (V, \sigma, \mu)$ be a PFLG. Choose an edge $(a, b)$ such that

$\mu'(a, b) = \big(\mu_1(a, b), \mu_2(a, b), \mu_3(a, b)\big)$ is the maximum in the set of all values of

$\mu'(a_i, b_i) = (\mu_1(a_i, b_i), \mu_2(a_i, b_i), \mu_3(a_i, b_i)) > 0$ and there exists some edge $(u, v)$ such that

$$\mu'(u, v) < \mu'(a, b).$$

Now we claim $\mu'(a, b)$ is a picture fuzzy bridge. From G if the edge $(a, b)$ is removed, then $H$ is a picture fuzzy subgraph, thus we have

$$\mu_H(a, b) < \mu'(a, b).$$

Therefore $(a, b)$ is a picture fuzzy bridge.

**Proposition 3.23:**

If G is a connected PFLG then there exists a strong picture fuzzy path between any pair of nodes.

**Proof:**

Let $G = (V, \sigma, \mu)$ be a connected PFLG and let $(a, b)$ be any pair of nodes. This implies $(a, b) = (\mu_1(a, b), \mu_2(a, b), \mu_3(a, b)) > 0$. Now choose any edge $(a, c)$ in $(a, b)$. If $\mu(a, c) = \mu'(a, c)$, then it is picture fuzzy strong. Otherwise choose any other edge, say $(a, d)$, which satisfies $\mu(a, d) = \mu'(a, d)$. By repeating this process, we can find a picture fuzzy path in $(a, b)$ in which all picture fuzzy arcs are strong.

**Proposition 3.24:**

Every PFLG has at least one weakest picture fuzzy arc.

**Proof:**

Let G be a PFLG and let $(a, b)$ be an edge of G such that the positive, neutral and negative memberships of this edge is maximum, maximum and minimum than all other edges of the picture fuzzy graph. If this edge is removed from G, it does reduce the picture fuzzy strength of any picture fuzzy path. That is, after its removal, PM, NM and NEM of $(a, b)$ in G is less, less and greater than the PM, NM and NEM of $(a, b)$ in the picture fuzzy labelling subgraph. This implies that the edge is neither a picture fuzzy bridge nor a strong picture fuzzy arc. Therefore, this must be the weakest picture fuzzy arc.

Now we define a Picture fuzzy labelling tree along with some of its properties.

**Definition 3.25:**

A PFL graph $G = (V, \sigma, \mu)$ is named a PFL tree, if it has PFL and a PF spanning subgraph $F = (V, \vartheta, \rho)$ which is a picture fuzzy tree, where for all PF arcs $(a, b)$ not in F, $\mu_1(a, b) < \rho_1^{\infty}(a, b)$, $\mu_2(a, b) < \rho_2^{\infty}(a, b)$, $\mu_3(a, b) > \rho_3^{\infty}(a, b)$.

**Theorem 3.26:**

If G is a PFL tree, then the arcs of PF spanning subgraph F are PF bridges of G.

**Proof:**

Let $G = (V, \sigma, \mu)$ be a PFL tree and $F = (V, \vartheta, \rho)$ be its PF spanning subgraph. Let $(a, b)$ bean arc in F. Then $\rho_1^{\infty}(a, b) < \mu_1(a, b) \leq \mu_1^{\infty}(a, b)$, $\rho_2^{\infty}(a, b) < \mu_2(a, b) \leq \mu_2^{\infty}(a, b)$ and $\rho_3^{\infty}(a, b) > \mu_3(a, b) \geq \mu_3^{\infty}(a, b)$, which implies that the picture fuzzy arc $(a, b)$ is a PF bridge of G. Since the picture fuzzy arc $(a, b)$ is arbitrary, the picture fuzzy arcs of F are PF bridges of G.

**Definition 3.27:**

A connected PFLG $G = (V, \sigma, \mu)$ is called a picture partial tree if G has a picture fuzzy spanning subgraph $F = (V, \vartheta, \rho)$ which is a picture fuzzy tree, where for all picture fuzzy arcs $(a, b)$ of G not in F, we have $PCONN_{1G}(a, b) > \mu_1(a, b)$, $PCONN_{2G}(a, b) > \mu_2(a, b)$, $PCONN_{3G}(a, b) < \mu_3(a, b)$.

If the above criterion is satisfied for all the components of G when it is not connected, then the PFLG is called a picture fuzzy partial forest.

**Example 3.28:**

Here is an example for a picture fuzzy partial tree. The graph G is the picture fuzzy partial tree since it has a picture fuzzy spanning tree F when the edge $(v_1, v_2)$ is removed. In the example below, $PCONN_G(v_1, v_2) = (0.2, 0.2, 0.2)$ and $\mu(v_1, v_2) = (0.1, 0.1, 0.3)$, thus G is a picture fuzzy partial tree.

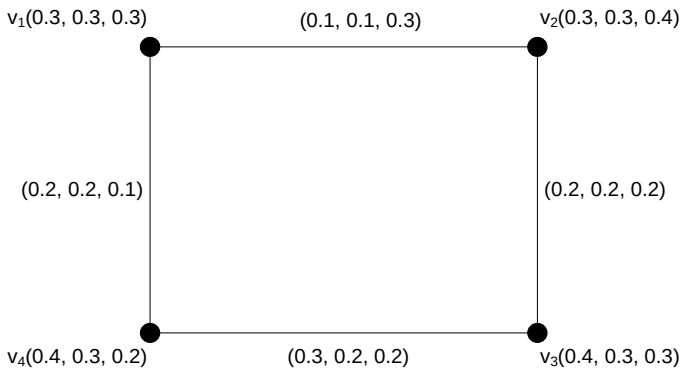


**Figure 7** Picture fuzzy partial tree

**Theorem 3.29:**

If there is at most one stronger picture fuzzy path among any two vertices of PFLG, then the PFLG should be a picture fuzzy partial forest.

**Proof:**

Let us assume that the PFLG is not a picture fuzzy partial forest. Then there will be a picture fuzzy cycle C in G with the criteria that $PCONN_{1G}(a,b) \leq \mu_1(a,b), PCONN_{2G}(a,b) \leq \mu_2(a,b), PCONN_{3G}(a,b) \geq \mu_3(a,b)$ for all edges $(a,b)$ of C. Thus, this edge is the strongest picture fuzzy path from $a\ to\ b$. But if we pick this edge to be the frailest edge of C, then the other edges of C will also be the strongest picture fuzzy path from a $to\ b$, which is a contradiction.

**Theorem 3.30:**

If G is a picture fuzzy partial tree but not a picture fuzzy tree, then there exists at least one edge for which the membership of edge is less than the strength of connectedness of that edge.

**Proof:**

Let $G = (V, \sigma, \mu)$ be a picture fuzzy partial tree. Then by the definition of picture fuzzy partial tree there is a picture fuzzy spanning tree F of G with the condition $\mu_1(a,b) < PCONN_{1G}(a,b),\ \mu_2(a,b) < PCONN_{2G}(a,b),\ \mu_3(a,b) > PCONN_{3G}(a,b)$ for edges $(a,b)$ in G not in F. Since G is not a picture fuzzy tree, there is at least one such picture fuzzy arc and thus the theorem is proved.

**Theorem 3.31:**

Let G be a picture fuzzy partial tree and F the spanning picture fuzzy tree. Then the arcs of F are the picture fuzzy partial bridges of G.

**Proof:**

Let $G = (V, \sigma, \mu)$ be a picture fuzzy partial tree and $(a,b)$ an arc in F. Since F is a picture fuzzy spanning tree of G, this arc is a unique picture fuzzy path from $a\ to\ b$ in F. The result is trivial if there is no other picture fuzzy path in G from $a\ to\ b$. If there is a picture fuzzy path from $a\ to\ b$ in G, then the picture fuzzy path will definitely have a picture fuzzy arc $(a,b)$ such that $PCONN_{1G}(a,b) > \mu_1(a,b), PCONN_{2G}(a,b) > \mu_2(a,b), PCONN_{3G}(a,b) < \mu_3(a,b)$. Then

it is shown that $(a, b)$ is not a frailest edge of any picture fuzzy cycle in G. Thus $(a, b)$ is a picture fuzzy partial bridge.

**APPLICATION:**

Everything in our human body is very important, to make us live a happily life. One of the important systems is Circulatory system in our body. There are two types of circulatory system, open & closed circulatory system. We all possess close circulatory system where blood flows through a closed nexus of blood vessels. This system contains Heart, Arteries, Veins and blood as their parts. The blood circulation system was discovered by William Harvey. The major function of human heart is pumping of blood which in one cycle pumps 70 ml of blood with normal heartbeat as 72 beats per minute. Next it has right & left atrium and right & left ventricle in the anterior and posterior part. Veins are the vessels which carries blood towards heart, which has impure blood (blood with carbon di oxide). The blood form lungs to left atrium is carried b Pulmonary vein. Artery is the vessel which carries blood from heart to body. Artery contains pure blood (blood with oxygen) except Pulmonary arteries which has impure blood. This Pulmonary artery transfers blood from right ventricle to lungs. The right part of heart carries impure blood and left part has pure blood.

Now we are going to represent this circulation system using a fuzzy graph model. Mammals all have double circulation in which blood crosses two times from heart. This is how the system works: Right atrium receives impure blood from body to right ventricle and into pulmonary artery and this sends blood to lungs for purification. After purification, pulmonary vein send blood to left atrium and then into left ventricle. Then the blood which is purified is circulated in the body for different organs. In the model given below, we have represented the circulation system of human body. When examining a human circulation system, through proper fuzzy labelling we can easily make the diagnosis easy.

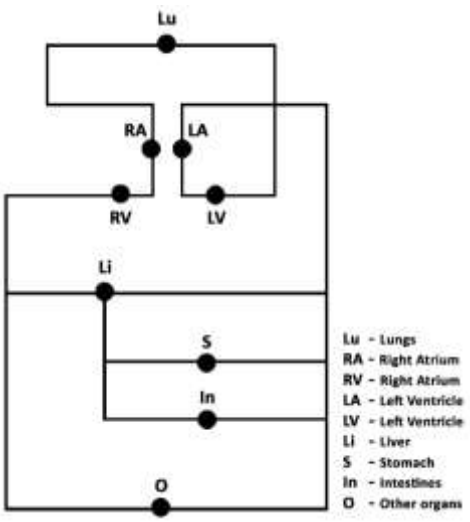


**Figure 8** Circulatory system model

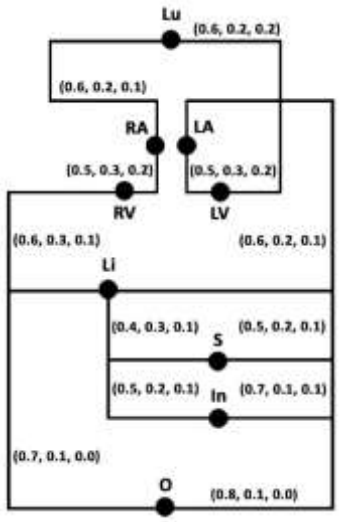


**Figure 9** Picture fuzzy labelling of circulatory system

In figure 9, we have just given an example or the picture fuzzy labelling. The first figure demonstrates the circulation system through a model, followed by the explanation of

how to label the model using picture fuzzy labelling. The values represent the blood flow through the veins. This model can be used for diagnosing the circulating system, since it can be represented as in the model and the blood flow can be measured. If it's below a particular level, then the corresponding treatment can be given for the patients. In future, we would like to develop this application for advancements.

**CONCLUSION:**

In this paper, we have introduced picture fuzzy labelling of graphs. And also investigated some important properties of picture fuzzy labelling graphs including picture fuzzy labelling cycle and PFL tree. Also an application of picture fuzzy graph labelling model for human circulatory system has been discussed which may be extended for many case studies. Furthermore, we would make a study on picture fuzzy magic labelling of graphs.

**REFERENCES:**


1. Zadeh, L.A., *Fuzzy sets,* Information and Control. 1965; Vol. 8, pp. 338-353.
2. Atanassov, K.T., *Intuitionstic fuzzy sets,* Fuzzy sets and systems. 1986; Vol. 20(1), pp. 87-96.
3. Smarandache, F., *A Unifying Field in Logics, Neutrosophic Logic: Neutrosophy, Neutrosophic Set, Neutrosophic Probability,* Rehoboth: American Research Press.1999.
4. Ajay, D., Manivel, M., & Aldring, J. (2019). Neutrosophic Fuzzy SAW Method and It's Application. *The International journal of analytical and experimental modal analysis,* 11(8), 881-887.
5. Ajay, D., & Aldring, J. (2019). A Decision Making Technique Based on Similarity Measure and Entropy of Bipolar Neutrosophic Sets. *The International journal of analytical and experimental modal analysis,* 11(9), 520-529.
6. Ajay, D., Broumi, S., & Aldring, J. (2020). An MCDM Method under Neutrosophic Cubic Fuzzy Sets with Geometric Bonferroni Mean Operator. *Neutrosophic Sets and Systems*, *32*(1), 13. DOI: 10.5281/zenodo.3723621.
7. Ajay, D., Aldring, J., Abirami, S., & Jeni Seles Martina, D. (2020). A SVTrN-number approach of multi-objective optimisation on the basis of simple ratio analysis based on MCDM method. *International Journal of Neutrosophic Science*, 5(1), 16-28.
8. Deli, I., & Şubaş, Y. (2017). Some weighted geometric operators with SVTrN-numbers and their application to multi-criteria decision making problems. *Journal of Intelligent & Fuzzy Systems*, *32*(1), 291-301.
9. Deli, I., & Öztürk, E. K. (2020). Two Centroid Point for SVTN-Numbers and SVTrN-Numbers: SVN-MADM Method. In *Neutrosophic Graph Theory and Algorithms* (pp. 279-307). IGI Global.
10. Cuong, B.C., Kreinovich, V., *Picture Fuzzy Sets, A new concept for computational intelligence problems,* In 2013 Third World Congress on Information and Communication Technologies, IEEE. 2013; pp. 1-6.

11. Cuong, B., *Picture Fuzzy Sets,* Journal of Computer Science in Cybernetics, 2014; Vol. 30(4), pp. 409-420.
12. Cuong, B.C., Huyen, P.T., Van Chien, P., Van Hai, P., *Some Fuzzy Inference Processes in Picture Fuzzy Systems,* In 2019 11th International Conference on Knowledge and Systems Engineering (KSE), IEEE. 2019; pp. 1-5.
13. Rosa, A., *On certain valuations of the vertices of a graph,* In: Proceeding of the International Symposium on Theory of Graphs, Rome, Italy. 1966.
14. Kaufmann, A., *Introduction to the theory of fuzzy subsets,* Academic Press.1975; Vol. 2.
15. Rosenfeld, A., *Fuzzy graphs, In Fuzzy sets and their applications to cognitive and decision processes,* Academic Press. 1975; pp. 77-95.
16. Gani, A.N., Akram, M., Subahashini, D.R., *Novel properties of fuzzy labelling graphs,* J.Math. 2014; 375135, pp. 1–6.
17. Ajay, D., Charisma, J.J., Chellamani, P., *Fuzzy Magic and Bi-Magic Labeling of Intuitionistic Path Graph,* International Journal of Recent Technology and Engineering, 2019; Vol. 8(4), pp. 11508-11512.
18. Ajay, D., and Chellamani, P. (2020)**.** Fuzzy magic labelling of Neutrosophic path and star graph, Advances in Mathematics: Scientific Journal, 9(8), p.6059–6070.
19. Zuo, C., Pal. A., Dey, A., *New Concepts of Picture Fuzzy Graphs with Application,* Mathematics. 2019; Vol. 7(5), pp. 470.

## References


1. Ajay, D., & Aldring, J. (2019). A Decision Making Technique Based on Similarity Measure and Entropy of Bipolar Neutrosophic Sets. The International journal of analytical and experimental modal analysis, 11(9), 520-529.
2. Ajay, D., Aldring, J., Seles Martina, D. J., & Abirami, S. A. (2020). A SVTrN-number approach of multi-objective optimisation on the basis of simple ratio analysis based on MCDM method. International Journal of Neutrosophic Science, 5(1), 16-28.
3. Ajay, D., and Chellamani, P. (2020). Fuzzy magic labelling of Neutrosophic path and star graph, Advances in Mathematics: Scientific Journal, 9(8), p.6059–6070.
4. Ajay, D., Broumi, S., & Aldring, J. (2020). An MCDM method under neutrosophic cubic fuzzy sets with geometric bonferroni mean operator. Neutrosophic Sets and Systems, 32, 187-202.
5. Ajay, D., Charisma, J. J., & Chellamani, P. (2019). Fuzzy Magic and Bi-magic Labelling of Intuitionistic Path Graph. International Journal of Recent Technology and Engineering, 8(4), 11508-11512.
6. Ajay, D., Manivel, M., & Aldring, J. (2019). Neutrosophic Fuzzy SAW Method and It's Application. The International journal of analytical and experimental modal analysis, 11, 881-887.
7. Atanassov, K.T. (1986); Intuitionstic fuzzy sets, Fuzzy sets and systemsVol. 20(1), pp.87-96.
8. Cường, B. C. (2014). Picture fuzzy sets. Journal of Computer Science and Cybernetics, 30(4), 409.
9. Cuong, B. C., & Kreinovich, V. (2013, December). Picture Fuzzy Sets-a new concept for computational intelligence problems. In 2013 third world congress on information and communication technologies (WICT 2013) (pp. 1-6). IEEE.
10. Cuong, B. C., Huyen, P. T., Van Chien, P., & Van Hai, P. (2019, October). Some Fuzzy Inference Processes in Picture Fuzzy Systems. In 2019 11th International Conference on Knowledge and Systems Engineering (KSE) (pp. 1-5). IEEE.
11. Deli, I., & Öztürk, E. K. (2020). Two Centroid Point for SVTN-Numbers and SVTrN-Numbers: SVN-MADM Method. In Neutrosophic Graph Theory and Algorithms (pp. 279-307). IGI Global.
12. Deli, I., & Şubaş, Y. (2017). Some weighted geometric operators with SVTrN-numbers and their application to multi-criteria decision making problems. Journal of Intelligent & Fuzzy Systems, 32(1), 291-301.
13. Kaufmann, A. (1975). Theory of fuzzy subsets: introduction to the. Fundamental theoretical elements. Academic Press.
14. Nagoor Gani, A., & Akram, M. (2014). Novel properties of fuzzy labeling graphs. Journal of Mathematics, 2014.
15. Rosa, A. (1966, July). Theory of graphs. In International Symposium, Rome, July (pp. 349-355).
16. Rosenfeld, A. (1975). Fuzzy graphs. In Fuzzy sets and their applications to cognitive and decision processes (pp. 77-95). Academic press.
17. Smarandache, F., A (1999).Unifying Field in Logics, Neutrosophic Logic: Neutrosophy, Neutrosophic Set, Neutrosophic Probability, Rehoboth: American Research Press.



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18. Zadeh, L.A., (1965) Fuzzy sets, Information and Control.; Vol. 8, pp. 338-353.
19. Zuo, C., Pal, A., & Dey, A. (2019). New concepts of picture fuzzy graphs with application. Mathematics, 7(5), 470.