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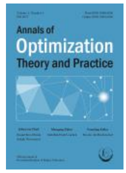

A new type of open set and its applications

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Abstract In this paper we introduce some new separation axioms by utilizing the notions of \alpha\omega-p-open sets and \alpha\omega-pre closure operator and the implication between the existing spaces are provided. Also as an application, we study some continuous functions and graph functions using this separation axioms. Basic theorems and properties are also investigated.

Keywords sober space; separation axioms; continuous function; graph function

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Abstract

In this paper we introduce some new separation axioms by utilizing the notions of $\alpha\omega$-p-open sets and $\alpha\omega$-pre closure operator and the implication between the existing spaces are provided. Also as an application, we study some continuous functions and graph functions using this separation axioms. Basic theorems and properties are also investigated.

Keywords. $\alpha\omega$-p-open, sober $(\alpha\omega, p)$-R_0, $D_{(\alpha\omega,p)}$-set, $(\alpha\omega, p)$-D_0, $(\alpha\omega, p)$-D_1, $(\alpha\omega, p)$-D_2.

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1. Introduction

The concept of preopen sets and precontinuous functions in topological spaces are introduced by A.S. Mashhour et al. [9]. Recently, M. Parimala et al. [13-15] introduced and studied the notion of $\alpha\omega$-open sets which are weaker than open sets. Since then, $\alpha\omega$-open sets have been widely used in order to introduce new spaces and functions.

In this paper, we introduce the notion of $\alpha\omega$-p-open sets and $\alpha\omega$-p-continuity in topological spaces. By utilizing these notions we introduce some weak separtion axioms. Also we show that some basic properties of $(\alpha\omega, p)$-T_i, $(\alpha\omega, p)$-D_i for $i = 0, 1, 2$ spaces and we ofer a new class of functions called $(\alpha\omega, p)$-continuous functions and a new notion of the graph of a function called an $(\alpha\omega, p)$-closed graph and investigate some of their fundamental properties.

2. Preliminaries

Let $A \subseteq X$, the closure of A and the interior of A will be denoted by $cl(A)$ and $int(A)$ respectively. A is regular open if $A = int(cl(A))$ and A is regular closed if its complement is regular open; equivalently A is regular closed if $A = cl(int(A))$, see [21].

Definition 2.1. A subset A of a space (X, τ) is called a

1. semi-open set [8] if $A \subseteq cl(int(A))$ and a semi-closed set if $int(cl(A)) \subseteq A$,
2. α-open set [10] if $A \subseteq int(cl(int(A)))$ and an α-closed set if $cl(int(cl(A))) \subseteq A$,
3. pre open set [9] if $A \subseteq int(cl(A))$ and pre closed set [10] if $cl(int(A)) \subseteq A$,
4. δ-open set [20] if for each $x \in A$, there exists a regular open set G such that $x \in G \subset A$ and
5. pre-regular p-open set if $A = pint(pcl(A))$.

The semi-closure (resp. α-closure) of a subset A of a space (X, τ) is the intersection of all semi-closed (resp. α-closed) sets that contain A and is denoted by $scl(A)$ (resp. $\alpha cl(A)$).

Definition 2.2. A subset A of a space (X, τ) is called a

1. a $\omega(=\widehat{g})$-*closed* set [17,19] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is *semi-open* in (X, τ). The complement of ω-*closed* set is called ω-*open* set and
2. a $\alpha\omega$-*closed* set [15] if $\omega cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-*open* in (X, τ). The complement of $\alpha\omega$-*closed* set is called $\alpha\omega$-*open* set.

Let (X, τ) be a space and let A be a subset of X. A is called $\alpha\omega$-closed set [15] if $\omega cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open set of (X, τ). The complement of an $\alpha\omega$-closed set is called $\alpha\omega$-open. The intersection of all $\alpha\omega$-closed (resp. δ-closed) sets containing A is called the $\alpha\omega$-closure (resp. δ-closure) of A and is denoted by $\mathrm{cl}_{\alpha\omega}(A)$ (resp. $\mathrm{cl}_{\delta}(A)$).

Definition 2.3. A subset A of a topological space (X, τ) is said to be δ-preopen

[9] if $A \subseteq \text{int}(\text{cl}_\delta(A))$. A family of all δ-preopen sets in a topological space (X, τ) is denoted by $\delta PO(X, \tau)$.

Definition 2.4. A function $f : X \to Y$ is called perfectly continuous [11] if for each open set $A \subset Y$, $f^{-1}(A)$ is open and closed in X.

Lemma 2.5. [6] If A and B are pre-regular p-open sets of the space X and$Y$, respectively, then $A \times B$ is a pre-regular p-open set of $X \times Y$.

Lemma 2.6. [6] If a space is submaximal, then any finite intersection of pre-regular p-open sets is pre-regular p-open.

3. $\alpha\omega$-p-open sets

Definition 3.1. A subset A of a topological space (X, τ) is said to be $\alpha\omega$-p-open if $A \subseteq \text{int}(\text{cl}_{\alpha\omega}(A))$.

The complement of an $\alpha\omega$-p-open set is said to be $\alpha\omega$-p-closed. The family of all $\alpha\omega$-p-open (resp. $\alpha\omega$-p-closed) sets in a topological space (X, τ) is denoted by $\alpha\omega PO(X, \tau)$ (resp. $\alpha\omega PC(X, \tau)$).

Definition 3.2. Let A be a subset of a topological space (X, τ). The intersection of all $\alpha\omega$-p-closed (resp. δ-preclosed) sets containing A is called the $\alpha\omega$-p-closure (resp. δ-preclosure [16]) of A and is denoted by $p\text{cl}_{\alpha\omega}(A)$ (resp. $p\text{cl}_\delta(A)$).

Definition 3.3. Let (X, τ) be a topological space. A subset U of X is called a $(\alpha\omega, p)$-neighbourhood of a point $x \in X$ if there exists an $\alpha\omega$-p-open set V such that $x \in V \subseteq U$.

Theorem 3.4. For the $\alpha\omega$-p-closure of subsets A, B in a topological space (X, τ), the following properties hold:

(1) A is $\alpha\omega$-p-closed in (X, τ) if and only if $A = p\text{cl}_{\alpha\omega}(A)$,

(2) If $A \subset B$, then $p\mathrm{cl}_{\alpha\omega}(A) \subset p\mathrm{cl}_{\alpha\omega}(B)$,

(3) $p\mathrm{cl}_{\alpha\omega}(A)$ is $\alpha\omega$-p-closed, that is $p\mathrm{cl}_{\alpha\omega}(A) = p\mathrm{cl}_{\alpha\omega}(p\mathrm{cl}_{\alpha\omega}(A))$ and

(4) $x \in p\mathrm{cl}_{\alpha\omega}(A)$ if and only if $A \cap V \neq \phi$ for every $V \in \alpha\omega PO(X, \tau)$ containing x.

Proof. It is obvious.

Theorem 3.5. For a family $\Big\{A_\beta : \beta \in \Delta\Big\}$ of subsets a topological space (X, τ), the following properties hold:

(1) $p\mathrm{cl}_{\alpha\omega}\Big(\cap \Big\{A_\beta : \beta \in \Delta\Big\}\Big) \subset \cap \Big\{p\mathrm{cl}_{\alpha\omega}(A_\beta) : \beta \in \Delta\Big\}$

(2) $p\mathrm{cl}_{\alpha\omega}\Big(\cup \Big\{A_\beta : \beta \in \Delta\Big\}\Big) \supset \cup \Big\{p\mathrm{cl}_{\alpha\omega}(A_\beta) : \beta \in \Delta\Big\}$

Proof.

(1) Since $\cap_{\beta\in\Delta} A_\beta \subset A_\beta$ for each $\beta \in \Delta$, by Theorem 3.4 we have $p\mathrm{cl}_{\alpha\omega}\Big(\cap_{\beta\in\Delta} A_\beta\Big) \subset p\mathrm{cl}_{\alpha\omega}(A_\beta)$ for each $\beta \in \Delta$ and hence $p\mathrm{cl}_{\alpha\omega}\Big(\cap_{\beta\in\Delta} A_\beta\Big) \subset \cap_{\beta\in\Delta} p\mathrm{cl}_{\alpha\omega}(A_\beta)$.

(2) Since $A_\beta \subset \cup_{\beta\in\Delta} A_\beta$ for each $\beta \in \Delta$, by Theorem 3.4 we have $p\mathrm{cl}_{\alpha\omega}(A_\beta) \subset p\mathrm{cl}_{\alpha\omega}\Big(\cup_{\beta\in\Delta} A_\beta\Big)$ for each $\beta \in \Delta$ and hence $\cup_{\beta\in\Delta} p\mathrm{cl}_{\alpha\omega}(A_\beta) \subset p\mathrm{cl}_{\alpha\omega}\Big(\cup_{\beta\in\Delta} A_\beta\Big)$.

Theorem 3.6. Every $\alpha\omega$-p-open set is preopen.

Proof. It follows from the Definitions.

The converse of the above Theorem need not be true by the following Example.

Example 3.7. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a, b\}\}$. Here $\{a, c\}$ is not $\alpha\omega$-p-open however it is preopen, since the $\alpha\omega$-p-open sets are $X, \phi, \{a\}, \{b\}, \{a, b\}$ and preopen sets are $X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}$.

Theorem 3.8.

(1) Every preopen set is δ-preopen [3].

(2) Every $\alpha\omega$-p-open is δ-preopen.

Proof. (2) It follows from (1) and Theorem 3.6.

Definition 3.9. A subset A of a topological space (X, τ) is called a $D_{(\alpha\omega,p)}$-set (resp. D_p-set [2,4], $D_{(\delta,p)}$-set [3]) if there are two $U, V \in \alpha\omega PO(X, \tau)$ (resp. $PO(X, \tau)$, $\delta PO(X, \tau)$) such that $U \neq X$ and $A = U - V$.

It is true that every $\alpha\omega$-p-open (resp. preopen) set U different from X is a $D_{(\alpha\omega,p)}$ -set (resp. D_p-set) if $A = U$ and $V = \phi$.

Definition 3.10. A topological space (X, τ) is said to be

(1) $(\alpha\omega, p)$-D_0 (resp. pre-D_0 [2,4], (δ, p)-D_0 [3]) if for any distinct pair of points x and y of X there exist a $D_{(\alpha\omega,p)}$-set (resp. D_p-set, $D_{(\delta,p)}$-set) of X containing x but not y or a $D_{(\alpha\omega,p)}$-set (resp. D_p-set, $D_{(\delta,p)}$-set) of X containing y but not x.

(2) $(\alpha\omega, p)$-D_1 (resp. pre-D_1 [2,4], (δ, p)-D_1 [3]) if for any distinct pair of points x and y of X there exist a $D_{(\alpha\omega,p)}$-set (resp. D_p-set, $D_{(\delta,p)}$-set) of X containing x but not y and a $D_{(\alpha\omega,p)}$-set (resp. D_p-set, $D_{(\delta,p)}$-set) of X containing y but not x.

(3) $(\alpha\omega, p)$-D_2 (resp. pre-D_2 [2,4], (δ, p)-D_2 [3]) if for any distinct pair of points x and y of X there exists disjoint $D_{(\alpha\omega,p)}$-set (resp. D_p-set, $D_{(\delta,p)}$-set) G and E of X containing x and y, respectively.

Definition 3.11. A topological space (X, τ) is said to be

(1) $(\alpha\omega, p)$-T_0 (resp. pre-T_0 [7,12], (δ, p)-T_0 [3]) if for any distinct pair of points x and y of X there exist an $\alpha\omega$-p-open (resp. preopen, δ-preopen) set U in

X containing x but not y or an $\alpha\omega$-p-open (resp. preopen, δ-open) set V in X containing y but not x.

(2) $(\alpha\omega, p)$-T_1 (resp. pre-T_1 [7,12], (δ, p)-T_1 [3]) if for any distinct pair of points x and y of X there exist an $\alpha\omega$-p-open (resp. preopen, δ-preopen) set U in X containing x but not y and an $\alpha\omega$-p-open (resp. preopen, δ-preopen) set V in X containing y but not x.

(3) $(\alpha\omega, p)$-T_2 (resp. pre-T_2 [7,12], (δ, p)-T_2 [3]) if for any distinct pair of points x and y of X there exist $\alpha\omega$-p-open (resp. preopen, δ-preopen) sets U and V in X containing x and y, respectively, such that $U \cap V = \phi$.

Remark 3.12.

(i) If (X, τ) is $(\alpha\omega, p)$-T_i, then it is $(\alpha\omega, p)$-$T_{i-1}, i = 1, 2$.

(ii) If (X, τ) is $(\alpha\omega, p)$-T_i, then it is $(\alpha\omega, p)$-$D_i, i = 0, 1, 2$.

(iii) If (X, τ) is $(\alpha\omega, p)$-D_i, then it is $(\alpha\omega, p)$-$D_{i-1}, i = 1, 2$.

(iv) If (X, τ) is $(\alpha\omega, p)$-D_i, then it is pre-$T_i, i = 0, 1, 2$.

By Remark 3.12 and [2, Remark 3.1], we have the following diagram.

$$
\begin{array}{ccccccccc}
(\alpha\omega,p)\text{-}T_2 & \longrightarrow & (\alpha\omega,p)\text{-}D_2 & \longrightarrow & \text{pre-}T_2 & \longrightarrow & (\delta,p)\text{-}T_2 & \longrightarrow & (\delta,p)\text{-}D_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\alpha\omega,p)\text{-}T_1 & \longrightarrow & (\alpha\omega,p)\text{-}D_1 & \longrightarrow & \text{pre-}T_1 & \longrightarrow & (\delta,p)\text{-}T_2 & \longrightarrow & (\delta,p)\text{-}D_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\alpha\omega,p)\text{-}T_0 & \longrightarrow & (\alpha\omega,p)\text{-}D_0 & \longrightarrow & \text{pre-}T_0 & \longrightarrow & (\delta,p)\text{-}T_2 & \longrightarrow & (\delta,p)\text{-}D_2
\end{array}
$$

Theorem 3.13. For a topological space (X, τ), the following properties hold: (X, τ) is $(\alpha\omega, p)$-D_1 if and only if it is $(\alpha\omega, p)$-D_2.

Proof. Sufficiency. This follows from Remark 3.12.

Necessity. Suppose X is a $(\alpha\omega, p)$-D_1. Then for each distinct pair $x, y \in X$, we have $D_{(\alpha\omega,p)}$-sets G_1 and G_2 such that $x \in G_1, y \notin G_1$; $y \in G_2, x \notin G_2$. Let $G_1 = U_1/U_2$, $G_2 = U_3/U_4$, where $U_1, U_2, U_3, U_4 \in \alpha\omega PO(X, \tau)$. From $x \notin G_2$ we have either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.

(1) $x \notin U_3$. From $y \notin G_1$ we have two subcases:
  (a) $y \notin U_1$. From $x \in U_1/U_2$ we have $x \in U_1/(U_2 \cup U_3)$ and from $y \in U_3/U_4$ we have $y \in U_3/(U_1 \cup U_4)$. It is easy to see that $(U_1/(U_2 \cup U_3)) \cap (U_3/(U_1 \cup U_4)) = \phi$.
  (b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1/U_2, y \in U_2$ and $(U_1/U_2) \cap U_2 = \phi$.

(2) $x \in U_3$ and $x \in U_4$. We have $y \in U_3/U_4, x \in U_4$ and $(U_3/U_4) \cap U_4 = \phi$.

From the discussion above we know that the space X is $(\alpha\omega, p)$-D_2.

Definition 3.14. A point $x \in X$ which has only X as the $(\alpha\omega, p)$-neighbourhood is called a $(\alpha\omega, p)$-neat point.

Theorem 3.15. If a topological spaces (X, τ) is $(\alpha\omega, p)$-D_1, then it has no $(\alpha\omega, p)$-neat point.

Proof. Since (X, τ) is $(\alpha\omega, p)$-D_1, so each point x of X is contained in a $D_{(\alpha\omega,p)}$-set $O = U/V$ and thus in U. By definition $U \neq X$. This implies that x is not a $(\alpha\omega, p)$-neat point.

Definition 3.16. A topological space (X, τ) is $(\alpha\omega, p)$-symmetric if x and y in X, $x \in p\mathrm{cl}_{\alpha\omega}(\{y\})$ implies $y \in p\mathrm{cl}_{\alpha\omega}(\{x\})$.

Theorem 3.17. For a topological space (X, τ), the following properties hold.

(1) If $\{x\}$ is $\alpha\omega$-p-closed for each $x \in X$, then (X, τ) is $(\alpha\omega, p)$-T_1.

(2) Every $(\alpha\omega, p)$-T_1 space is $(\alpha\omega, p)$-symmetric.

Proof.

(1) Suppose $\{p\}$ is $\alpha\omega$-p-closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X/\{x\}$. Hence $X/\{x\}$ is an $\alpha\omega$-p-open set contained in y but not containing x. Similarly $X/\{y\}$ is an $\alpha\omega$-p-open set contained in x but not containing y. Accordingly X is a $(\alpha\omega, p)$-T_1 space.

(2) Suppose that $y \notin \text{pcl}_{\alpha\omega}(\{x\})$. Then, since $x \neq y$, there exists an $\alpha\omega$-p-open set U containing x such that $y \notin U$ and hence $x \notin \text{pcl}_{\alpha\omega}(\{y\})$. This shows that $x \in \text{pcl}_{\alpha\omega}(\{y\})$ implies $y \in \text{pcl}_{\alpha\omega}(\{x\})$. Therefore (X, τ) is $(\alpha\omega, p)$-symmetric.

Definition 3.18. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\alpha\omega$-precontinuous if for each $x \in X$ and each $\alpha\omega$-p-open set V containing $f(x)$, there is an $\alpha\omega$-p-open set U ix X containing x such that $f(U) \subseteq V$.

Theorem 3.19. If $f : (X, \tau) \to (Y, \sigma)$ is an $\alpha\omega$-precontinuous surjective function and E is a $D_{(\alpha\omega,p)}$-set in Y, then the inverse image $f^{-1}(E)$ is a $D_{(\alpha\omega,p)}$-set in X.

Proof. Let E be a $D_{(\alpha\omega,p)}$ set in Y. Then there are $\alpha\omega$-p-open sets U_1 and U_2 in Y such that $E = U_1/U_2$ and $U_1 \neq Y$. By the $\alpha\omega$-precontinuity of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are $\alpha\omega$-p-open in X. Since $U_1 \neq Y$, we have $f^{-1}(U_1) \neq X$. Hence $f^{-1}(E) = f^{-1}(U_1)/f^{-1}(U_2)$ is a $D_{(\alpha\omega,p)}$-set.

Theorem 3.20. If (Y, σ) is $(\alpha\omega, p)$-D_1 and $f : (X, \tau) \to (Y, \sigma)$ is an $\alpha\omega$-precontinuous bijection, then (X, τ) is $(\alpha\omega, p)$-D_1.

Proof. Suppose that Y is a $(\alpha\omega, p)$-D_1 space. Let x and y be any pair of distinct points in X. Since F is injective and Y is $(\alpha\omega, p)$-D_1, there exist $D_{(\alpha\omega,p)}$-sets G_x and G_y of Y containing $f(x)$ and $f(y)$, respectively, such that $f(y) \notin G_x$ and $f(x) \notin G_y$. By Theorem 3.19, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are $D_{(\alpha\omega,p)}$-sets in X containing x and y, respectively, such that $y \notin f^{-1}(G_x)$ and $x \notin f^{-1}(G_y)$. This implies that X is a $(\alpha\omega, p)$-D_1 space.

Thoerem 3.21. A topological space (X, τ) is $(\alpha\omega, p)$-D_1 if and only if for each pair of distinct points $x, y \in X$, there exists an $\alpha\omega$-precontinuous surjective function $f : (X, \tau) \to (Y, \sigma)$ such that $f(x)$ and $f(y)$ are distinct, where (Y, σ) is a $(\alpha\omega, p)$-D_1 space.

Proof. Necessity. For every pair of distinct points of X, it suffices to take the identity function on X.

Sufficiency. Let x and y be any pair of distinct points in X. By hypothesis there exists an $\alpha\omega$-precontinuous, surjective function f of a space X onto $(\alpha\omega, p)$-D_1 space Y such that $f(x) \neq f(y)$. By Theorem 3.13, there exist disjoint $D_{(\alpha\omega,p)}$-sets G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is $\alpha\omega$-precontinuous and surjective, by Theorem 3.20, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint $D_{(\alpha\omega,p)}$-sets in X containing x and y, respectively, hence by Theorem 3.13, X is a $(\alpha\omega, p)$-D_1 space.

4. Sober $(\alpha\omega, p)$-R_0 spaces

Definition 4.1. Let A be a subset of a topological space (X, τ). The $\alpha\omega$-prekernel of A, denoted by $p\ker_{\alpha\omega}(A)$ is defined to be the set $p\ker_{\alpha\omega}(A) = \cap\Big\{U \in \alpha\omega PO(X, \tau) : A \subseteq U\Big\}$.

Lemma 4.2. Let (X, τ) be a topological space and $x \in X$. Then $p\ker_{\alpha\omega}(A) = \Big\{x \in X : p\mathrm{cl}_{\alpha\omega}(\{x\}) \cap A \neq \phi\Big\}$.

Proof. Let $x \in p\ker_{\alpha\omega}(A)$ and suppose $p\mathrm{cl}_{\alpha\omega}(\{x\}) \cap A = \phi$. Hence $x \notin X/p\mathrm{cl}_{\alpha\omega}(\{x\})$ which is an $\alpha\omega$-p-open set containing A. This is absurd, since $x \in p\ker_{\alpha\omega}(A)$. Consequently, $p\mathrm{cl}_{\alpha\omega}(\{x\}) \cap A \neq \phi$. Next, let x be such that $p\mathrm{cl}_{\alpha\omega}(\{x\}) \cap A \neq \phi$ and suppose that $x \notin p\ker_{\alpha\omega}(A)$. Then, there exists an $\alpha\omega$-p-open set D containing A and $x \notin D$. Let $y \in p\mathrm{cl}_{\alpha\omega}(\{x\}) \cap A$. Hence, D is a $(\alpha\omega, p)$-neighbourhood of y which does not containing x. By this contradiction $x \in p\ker_{\alpha\omega}(A)$ and the claim is shown.

Definition 4.3. A topological space (X, τ) is said to be sober $(\alpha\omega, p)$-R_0 (resp.

sober (δ, p)-R_0 [3]) if $\cap_{x\in X} p\mathrm{cl}_{\alpha\omega}(\{x\}) = \phi$ (resp. $\cap_{x\in X} p\mathrm{cl}_{\delta}(\{x\}) = \phi$).

Theorem 4.4. Every sober $(\alpha\omega, p)$-R_0 space is sober (δ, p)-R_0 space.

Proof. Let (X, τ) be a sober $(\alpha\omega, p)$-R_0 space, then $\cap_{x\in X} p\mathrm{cl}_{\alpha\omega}(\{x\}) = \phi$. Therefore, $\cap_{x\in X} p\mathrm{cl}_{\delta}(\{x\}) = \phi$.

Theorem 4.5. A topological space (X, τ) is sober $(\alpha\omega, p)$-R_0 if and only if $p\mathrm{ker}_{\alpha\omega}(\{x\}) \neq X$ for every $x \in X$.

Proof. Suppose that the space (X, τ) be sober $(\alpha\omega, p)$-R_0. Assume that there is a point y in X such that $p\mathrm{ker}_{\alpha\omega}(\{y\}) = X$. Let x be any point of X. Then $x \in V$ for every $\alpha\omega$-p-open set V containing y and hence $y \in p\mathrm{cl}_{\alpha\omega}(\{x\})$ for any $x \in X$. This implies that $y \in \cap_{x\in X} p\mathrm{cl}_{\alpha\omega}(\{x\})$. But this is a contradiction. Now assume that $p\mathrm{ker}_{\alpha\omega}(\{x\}) \neq X$ for every $x \in X$. If there exists a point of X. This implies that the space X is the unique $\alpha\omega$-preopen set containing y. Hence $p\mathrm{ker}_{\alpha\omega}(\{y\}) = X$ which is a contradiction. Therefore (X, τ) is sober $(\alpha\omega, p)$-R_0 space.

Definition 4.6. A function $f : (X, \tau) \to (Y, \sigma)$ is called pre $\alpha\omega$-p-closed if the image of every $\alpha\omega$-p-closed subset of X is $\alpha\omega$-p-closed in Y.

Theorem 4.7. If $f : (X, \tau) \to (Y, \sigma)$ is an injective pre $\alpha\omega$-p-closed function and X is sober $(\alpha\omega, p)$-R_0, then Y is sober $(\alpha\omega, p)$-R_0.

Proof. Since X is sober $(\alpha\omega, p)$-R_0, $\cap_{x\in X} p\mathrm{cl}_{\alpha\omega}(\{x\}) = \phi$. Since f is a pre $\alpha\omega$-p-closed injection, we have

$$\begin{aligned} \phi &= f\Big(\cap_{x\in X}\, p\mathrm{cl}_{\alpha\omega}(\{x\})\Big) \\ &= \cap_{x\in X} f\Big(p\mathrm{cl}_{\alpha\omega}(\{x\})\Big) \\ &\supseteq \cap_{x\in X} p\mathrm{cl}_{\alpha\omega}(\{f(x)\}) \\ &\supseteq \cap_{y\in Y} p\mathrm{cl}_{\alpha\omega}(\{y\}). \end{aligned}$$

Therefore, Y is sober $(\alpha\omega, p)$-R_0.

Theorem 4.8. If a topological space X is sober $(\alpha\omega, p)$-R_0 and Y is any topological space, then the product $X \times Y$ is sober $(\alpha\omega, p)$-R_0.

Proof. We show that $\cap_{(x,y)\in X\times Y} p\mathrm{cl}_{\alpha\omega}(\{(x,y)\}) = \phi$. We have

$$\begin{aligned} \cap_{(x,y)\in X\times Y} p\mathrm{cl}_{\alpha\omega}(\{(x,y)\}) &\subseteq \cap_{(x,y)\in X\times Y} \Big(p\mathrm{cl}_{\alpha\omega}(\{x\}) \times p\mathrm{cl}_{\alpha\omega}(\{y\})\Big) \\ &= \cap_{x\in X} p\mathrm{cl}_{\alpha\omega}(\{x\}) \times \cap_{y\in Y} p\mathrm{cl}_{\alpha\omega}(\{y\}) \\ &\subseteq \phi \times Y \\ &= \phi. \end{aligned}$$

5. $(\alpha\omega, p)$-continuous functions and $(\alpha\omega, p)$-closed graphs

Definition 5.1.A function $f : X \to Y$ is said to be $(\alpha\omega, p)$-continuous if for every open set V of Y, $f^{-1}(V)$ is $(\alpha\omega, p)$-open in X.

Theorem 5.2. The following are equivalent for a function $f : X \to Y$:

(i) f is $(\alpha\omega, p)$-continuous,

(ii) The inverse image of every closed set in Y is $(\alpha\omega, p)$-closed in X,

(iii) For each subset A of X, $f(\alpha\omega cl_p(A)) \subset cl(f(A))$,

(iv) For each subset B of Y, $\alpha\omega cl_p(f^{-1}(B)) \subset f^{-1}(cl(B))$.

Proof. (i) $\Leftrightarrow$ (ii): Obvious.

(iii) $\Leftrightarrow$ (iv): Let B be any subset of Y. Then by (iii), we have $f(\alpha\omega cl_p(f^{-1}(B))) \subset cl(f(f^{-1}(B))) \subset cl(B)$. This implies $\alpha\omega cl_p(f^{-1}(B)) \subset f^{-1}(f(\alpha\omega cl_p(f^{-1}(B)))) \subset f^{-1}(cl(B))$.

Conversely, let $B = f(A)$ where A is a subset of X. Then, by (iv), we have, $\alpha\omega cl_p(A) \subset \alpha\omega cl_p(f^{-1}(f(A))) \subset f^{-1}(cl(f(A)))$. Thus, $f(\alpha\omega cl_p(A)) \subset cl(f(A))$.

(ii) $\Rightarrow$ (iv): Let $B \subset Y$. Since $f^{-1}(cl(B))$ is $(\alpha\omega, p)$-closed and $f^{-1}(B) \subset f^{-1}(cl(B))$, then $\alpha\omega cl_p(f^{-1}(B)) \subset f^{-1}(cl(B))$.

(iv) $\Rightarrow$ (ii): Let $K \subset Y$ be a closed set. By (iv), $\alpha\omega cl_p(f^{-1}(K)) \subset f^{-1}(cl(K)) = f^{-1}(K)$. Thus, $f^{-1}(K)$ is $(\alpha\omega, p)$-closed.

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\}$ of the product space $X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 5.3. For a function $f : X \to Y$, the graph $G(f) = \{(x, f(x)) : x \in X\}$ is said to be $(\alpha\omega, p)$-closed if for each $(x, y) \in X \times Y - G(f)$, there exist $U \in \alpha\omega PO(X, x)$ and an open set V of Y containing y such that $(U \times V) \cap G(f) = \phi$.

Lemma 5.4. Let $f : X \to Y$ be a function. Then the graph $G(f)$ is $(\alpha\omega, p)$-closed in $X \times Y$ if and only if for each point $(x, y) \in X \times Y - G(f)$, there exist a $(\alpha\omega, p)$-open set U and an open set V containing x and y, respectively, such that $f(U) \cap V = \phi$.

Proof. It follows readily from the above definition.

Theorem 5.5. If $f : X \to Y$ is an injective function with the $(\alpha\omega, p)$-closed graph, then X is $(\alpha\omega, p)$-T_1.

Proof. Let x and y be two distinct points of X. Then $f(x) \neq f(y)$. Thus there exist an $(\alpha\omega, p)$-open set U and an open set V containing x and $f(y)$, respectively, such that $f(U) \cap V = \phi$. Therefore $y \notin U$ and it follows that X is $(\alpha\omega, p)$-T_1.

Recall that a space X is said to be T_1 if for each pair of distinct points x and y of X, there exist an open set U containing x but not y and an open set V containing y but not x.

Theorem 5.6. If $f : X \to Y$ is an surjective function with the $(\alpha\omega, p)$-closed graph, then Y is T_1.

Proof. Let y_1 and y_2 be two distinct points of Y. Since f is surjective, there exist a point x in X such that $f(x) = y_2$. Therefore $(x, y_1) \notin G(f)$. By lemma 5.4., there exist an $(\alpha\omega, p)$-open set U and an open set V containing x and y_1, respectively, such that $f(U) \cap V = \phi$. It follows that $y_2 \notin V$. Hence Y is T_1.

Definition 5.7. A function $f : X \to Y$ is said to be $(\alpha\omega, p)$-W-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an $(\alpha\omega, p)$-open set U in X containing x such that $f(U) \subset cl(V)$.

Theorem 5.8. If $f : X \to Y$ is $(\alpha\omega, p)$-W-continuous and Y is Hausdorff, then $G(f)$ is $(\alpha\omega, p)$-closed.

Proof. Suppose that $(x, y) \notin G(f)$, then $f(x) \neq y$. By the fact that Y is Hausdorff, there exist open sets W and V such that $f(x) \in W$, $y \in V$ and $V \cap W = \phi$. It follows that $cl(W) \cap V = \phi$. Since f is $(\alpha\omega, p)$-W-continuous, there exists $U \in \alpha\omega PO(X, x)$ such that $f(U) \subset cl(W)$. Hence, we have $f(U) \cap V = \phi$. This means that $G(f)$ is $(\alpha\omega, p)$-closed.

Corollary 5.9. If $f : X \to Y$ is $(\alpha\omega, p)$-W-continuous and Y is Hausdorff, then $G(f)$ is $(\alpha\omega, p)$-closed in $X \times Y$.

Definition 5.10. A subset A of a space X is said to be $(\alpha\omega, p)$-compact relative to X if every cover of A by $(\alpha\omega, p)$-open sets of X has a finite subcover.

Theorem 5.11. Let $f : X \to Y$ have a $(\alpha\omega, p)$-closed graph. If K is $(\alpha\omega, p)$-compact relative to X, then $f(K)$ is closed in Y.

Proof. Suppose that $y \notin f(K)$. For each $x \in K$, $f(x) \neq y$. By lemma 5.4., there exist $U_x \in \alpha\omega PO(X, x)$ and an open neighbourhood V_x of y such that $f(U_x) \cap V_x = \phi$. The family $\{U_x : x \in K\}$ is a cover of K by $(\alpha\omega, p)$-open sets of X and there exists a fnite subset K_0 of K such that $K \subset \cup\{U_x : x \in K_0\}$. Put $V = \cap\{V_x : x \in K_0\}$. Then V is an open neighbourhood of y and $f(K) \cap V = \phi$. This means that $f(K)$ is closed in Y.

Theorem 5.12. If $f : X \to Y$ has an $(\alpha\omega, p)$-closed graph $G(f)$ and $g : Y \to Z$ is a perfectly continuous function, then the set $\{(x, y) : f(x) = g(y)\}$ is $(\alpha\omega, p)$-

closed in $X \times Y$.

Proof. Let $A = \{(x, y) : f(x) = g(y)\}$ and $(x, y) \in (X \times Y) - G(f)$. Since f has an $(\alpha\omega, p)$-closed graph $G(f)$, there exist an $(\alpha\omega, p)$-open set U and an open set V containing x and $g(y)$, respectively, such that $f(U) \cap V = \phi$. This implies that there exists a pre-regular p-open set N containing x such that $N \subset U$ and $f(N) \cap V = \phi$. Since g is a perfectly continuous function, then there exist an open and closed set G containig y such that $g(G) \subset V$. We have $f(U) \cap g(G) = \phi$. This implies that $(N \times G) \cap A = \phi$. Since $N \times G$ is pre-regular p-open, then $(x, y) \notin \alpha\omega cl_p(A)$. Thus, E is $(\alpha\omega, p)$-closed in $X \times Y$.

Corollary 5.13. If $f : X \to Z$ is an $(\alpha\omega, p)$-continuous function and $g : Y \to Z$ is a perfectly continuous function and Z is Hausdorff, then the set $\{(x, y) : f(x) = g(y)\}$ is $(\alpha\omega, p)$-closed in $X \times Y$.

Proof. It follows from Corollary 5.9 and Theorem 5.12.

Theorem 5.14. If $f : X \to Y$ is an $(\alpha\omega, p)$-continuous function and Y is Hausdorff, then the set $\{(x, y) \in X \times Y : f(x) = f(y)\}$ is $(\alpha\omega, p)$-closed in $X \times X$.

Proof. Let $\{(x, y) : f(x) = f(y)\}$ and let $\{(x, y) \in (X \times Y) - A\}$. It follows tat $f(x) \neq f(y)$. Since Y is Hausdorff, there exist open set U and V containing $f(x)$ and f(y), respectively, such that $U \cap V = \phi$. Since f is $(\alpha\omega, p)$-continuous, there exist pre-regular p-open set in $X \times X$ containing (x, y). Hence, A is $(\alpha\omega, p)$-closed in $X \times X$.

Definition 5.15. A function $f : X \to Y$ is called contra $(\alpha\omega, p)$-open if the image of every $(\alpha\omega, p)$-open set in X is closed in Y.

Theorem 5.16. If $f : X \to Y$ is a contra $(\alpha\omega, p)$-open funcion such that the inverse image of each opoint of Y is $(\alpha\omega, p)$-closed, then f has an $(\alpha\omega, p)$-closed graph $G(f)$.

Proof. Let $(x, y) \in X - G(f)$. We have $x \notin f^{-1}(y)$. Since $f^{-1}(y)$ is $(\alpha\omega, p)$-closed, there exists a pre-regular p-open set A containing x such that $A \cap f^{-1}(y) = \phi$. Since, f is contra $(\alpha\omega, p)$-open, then $f(A)$ is closed. This implies that there exist an open set B in Y containing y such that $f(A) \cap B = \phi$. Hence, f has an $(\alpha\omega, p)$-closed graph $G(f)$.

Theorem 5.17. If $f : X \to Y$ has an $(\alpha\omega, p)$-closed graph $G(f)$, then for each $x \in X$, $\{f(x)\} = \cap_{x\in A\in\alpha\omega PO(X,\tau)} cl(f(A))$.

Proof. Suppose that $y \neq f(x)$ and $y \in \cap_{x\in A\in\alpha\omega PO(X,\tau)} cl(f(A))$. Then $y \in cl(f(A))$ for each $x \in A \in \alpha\omega PO(X, \tau)$. This implies that for each open set B containing y, $B \cap f(A) \neq \phi$. Since $(x, y) \notin G(f)$ and $G(f)$ is an $(\alpha\omega, p)$-closed graph, this is a contradiction.

Definition 5.18. A function $f : X \to Y$ is called an $(\alpha\omega, p)$-open if the image of every $(\alpha\omega, p)$-open set in X is open in Y.

Theorem 5.19. If $f : X \to Y$ is a surjective $(\alpha\omega, p)$-open function with an $(\alpha\omega, p)$-closed graph $G(f)$, then Y is T_2.

Proof. Let y_1 and y_2 be any two distinct points of Y. Since f is surjective $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) - G(f)$. This implies that there exist an $(\alpha\omega, p)$-open set A of X and an open set B of Y such that $(x, y_2) \in (A\times B)$ and $(A \times B) \cap G(f) = \phi$. We have $f(A) \cap B = \phi$. Since f is $(\alpha\omega, p)$-open, then $f(A)$ is open such that $f(x) = y_1 \in f(A)$. Thus, Y is T_2.

Theorem 5.20. If $f : X \to Y$ is an $(\alpha\omega, p)$-continuous injective function and Y is T_2, then X is $(\alpha\omega, p)$-T_2.

Proof. Let x and y in X be any pair of distinct points, then there exist disjoint open sets A and B in Y such that $f(x) \in A$ and $f(y) \in B$. Since f is $(\alpha\omega, p)$-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are $(\alpha\omega, p)$-open in X containing x and y respectively, we have $f^{-1}(A) \cap f^{-1}(B) = \phi$. Thus, X is $(\alpha\omega, p)$-T_2.

Theorem 5.21. If $f, g : X \to Y$ are $(\alpha\omega, p)$-continuous functions, X is submaximal and Y is Hausdorff, then the set $\{x \in X : f(x) = g(x)\}$ is $(\alpha\omega, p)$-closed in X.

Proof. Let $A = \{x \in X : f(x) = g(x)\}$. Take $x \in X - A$. We have $f(x) \neq g(x)$. Since Y is Hausdorff, then there exist open sets U and V in Y containing $f(x)$ and $g(x)$, respectively, such that $U \cap V = \phi$. Since f and g are $(\alpha\omega, p)$-continuous, then $f^{-1}(U)$ and $g^{-1}(V)$ are $(\alpha\omega, p)$-open in X with $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$. Then there exist pre-regular p-open sets G and H such that $x \in G \subset f^{-1}(U)$ and $x \in H \subset g^{-1}(V)$. Take $K = G \cap H$. By lemma 2.6, K is pre-regular p-open. Thus, $f(K) \cap g(K) = \phi$ and hence $x \notin \alpha\omega cl_p(A)$. This shows that A is $(\alpha\omega, p)$-closed in X.

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