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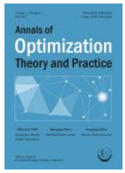

Numerical study of a mathematical model of disease caused by water pollution

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Abstract In Typhoid fever is one of the most common diseases caused by food and water pollution and one of the environmental problems currently in the industrialized world. The infectious disease, like other diseases, including AIDS, hepatitis, etc., is modeled as a nonlinear differential equation system. In this paper, a matrix-Galerkin method is introduced for the numerical study of the mathematical model of Typhoid fever. This method in based on matrix-algabraic computation of Galerkin method that converts the model to algebraic equations. The main purpose is to find an approximate solution with a simple algorithm to determine the population behavior in the Typhoid model. The results of the method show the accuracy and efficiency of the method.

Keywords Matrix-Galerkin method; Typhoid model; Convergence analysis

1. Introduction

The Typhoid fever is an infectious disease caused by bacteria "Salmonella typhi", an infectious disease that spreads through contaminated water and food, and usually occurs with high fever, diarrhea, anorexia, and headache Cvjetanović *et al.* (2014) and Hyman (2006). The typhoid bacteria in muddy water survive for up to a month and in ice for up to three months; it is destroyed by the heat of 60-100 ° C, and especially sunlight quickly destroys the bacteria. It is resistant to drought for up to two months. The infectious disease may become sporadic or epidemic. In the autumn and summer, it becomes more pervasive, with epidemics, especially in communities like barracks and schools. The disease in Iran is abundant in all seasons, but more in summer and autumn. In the time of war, all these conditions are prevalent. Newcomers entering the infected areas are more likely to develop than indigenous people. Mathematical models are used to analyze the dynamics of the disease such as AIDS, malaria, hepatitis, and so on. In this paper, we investigate the numerical results of a mathematical model of typhoid disease, as a

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nonlinear differential equation system. The Matrix-Galerkin method is a spectral-numerical method for approximating physical problems and mathematical models in which solutions of problems with standard foundations are approximated by finite dimension.

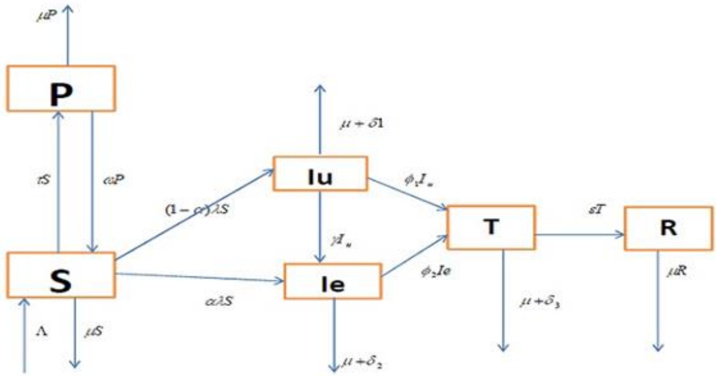

Figure 1. Typhoid model

2. Typhoid fever model

In The following system of differential equations

$$\begin{aligned}
\frac{dP}{dt} &= \tau S - (\mu + \omega)P - \mu S \\
\frac{dS}{dt} &= \Lambda - S(\beta_1 Iu + \beta_2 Ie + \beta_3 T) + \omega P - (\tau + \mu)S \\
\frac{dIu}{dt} &= (1 - \alpha_1)S(\beta_1 Iu + \beta_2 Ie + \beta_3 T) - (\mu + \delta_1 + \varphi_1 + \gamma)Iu \\
\frac{dIe}{dt} &= \alpha S(\beta_1 Iu + \beta_2 Ie + \beta_3 T) - (\mu + \delta_2 + \varphi_2)Ie + \gamma Iu \\
\frac{dT}{dt} &= \varphi_1 \mathrm{Iu} + \varphi_2 \mathrm{Ie} - (\mu + \delta_3 + \varepsilon)T \\
\frac{dR}{dt} &= \varepsilon \mathrm{T} - \mu \mathrm{R}
\end{aligned} \tag{1}$$

With initial conditions:

$\mathrm{P}(0) = P0, \mathrm{S}(0) = S0, \mathrm{Ie}(0) = IE, \mathrm{Iu}(0) = IU, \mathrm{T}(0) = T0, \mathrm{R}(0) = R0.$

Is called "Typhoid fever model". Moatlhodl and Gosaamang (2017).

Theorem 1: (Existence and uniqueness solution**)**

For the initial value problem

$$x'(t) = f\big(t, x(t)\big), \quad x(0) = x_0 \tag{2}$$

If the function f, satisfy in Lipschitz condition i.e.,

$$\left|f\big(t,x(t)\big)-f\big(t,y(t)\big)\right| \leq \boldsymbol{L}|x-y|,$$

Or, bounded partial derivative respect to $x(t)$, i.e.,

$$\left|\frac{\partial f}{\partial x}\right| < \infty$$

Then, the problem (2) has a unique continuous solution. (See Simmons (1972), for more detail).

In our model (1), we assume that

$$\begin{aligned}
A1 &= \tau S-(\mu+\omega)P-\mu S \\
A2 &= \Lambda-S(\beta_1 Iu+\beta_2 Ie+\beta_3 T)+\omega P-(\tau+\mu)S \\
A3 &= (1-\alpha_1)S(\beta_1 Iu+\beta_2 Ie+\beta_3 T)-(\mu+\delta_1+\varphi_1+\gamma)Iu \\
A4 &= \alpha S(\beta_1 Iu+\beta_2 Ie+\beta_3 T)-(\mu+\delta_2+\varphi_2)Ie+\gamma Iu \\
A5 &= \varphi_1 \mathrm{Iu}+\varphi_2 \mathrm{Ie}-(\mu+\delta_3+\varepsilon)T \\
A6 &= \varepsilon \mathrm{T}-\mu \mathrm{RR}
\end{aligned} \tag{3}$$

Now, basd on Thereom (1), for all terms $A1, A2, \dots, A6$, have the bounded partial derivate respect to independent variables $P, S, \dots, R$. For example,

$$\frac{\partial A1}{\partial P} = -(\mu+\omega) < 0, \qquad \frac{\partial A1}{\partial S} = \tau < 0$$
$$\frac{\partial A1}{\partial Ie} = 0, \qquad \frac{\partial A1}{\partial Iu} = 0$$
$$\frac{\partial A1}{\partial T} 0, \qquad \frac{\partial A1}{\partial R} = 0$$

Then, we can conclude that the model (1) has a unique solution.

Main result: Matrix-Galerkin method

Let

$$\begin{aligned}
&\mathrm{P_N(t)} = \sum_{k=0}^{N} \mathrm{p_k t^k}, \quad \mathrm{S_N(t)} = \sum_{k=0}^{N} \mathrm{s_k t^k}, \mathrm{Iu_N(t)} = \sum_{k=0}^{N} \mathrm{u_k t^k} \\
&\mathrm{Ie_N(t)} = \sum_{k=0}^{N} \mathrm{e_k t^k}, \quad \mathrm{T_N(t)} = \sum_{k=0}^{N} \mathrm{T_k t^k}, \mathrm{R_N(t)} = \sum_{k=0}^{N} \mathrm{r_k t^k}
\end{aligned} \tag{4}$$

Are the approximation for all function $\mathrm{P(t)}, S(t), Iu(t), Ie(t), T(t)$ *and* $R(t)$ in finite dimenstional space $X_N = [1, \mathrm{t}, \dots, t^N]$. By setting all unknown coefficints in vector form, we obtain

$$\begin{aligned}
&\mathrm{P} = [\mathrm{p_0, p_1, \dots, p_N}], \qquad \mathrm{S} = [\mathrm{s_0, s_1, \dots, s_N}], \qquad \mathrm{Iu} = [\mathrm{u_0, u_1, \dots, u_N}] \\
&\mathrm{Ie} = [\mathrm{e_0, e_1, \dots, e_N}], \qquad \mathrm{T} = [\mathrm{T_0, T_1, \dots, T_N}], \qquad \mathrm{R} = [\mathrm{r_0, r_1, \dots, r_N}]
\end{aligned} \tag{5}$$

That, yelids

$\mathrm{P_N(t)} = \mathbf{P}.X_N,\ \ \mathrm{S_N(t)} = \mathbf{S}.X_N, \mathrm{Iu_N(t)} = \mathbf{Iu}.X_N, \mathrm{Ie(t)} = \mathbf{Ie}.X_N,\ \ \mathrm{T_N(t)} = \mathbf{T}.X_N, \mathrm{R_N(t)} = \mathbf{R}.X_N.$

Theorem 2:

Assume that

$$M=\begin{bmatrix} 0 & \cdots & 0 \\ 1 & \ddots & \vdots \\ 0 & \cdots N & 0 \end{bmatrix}, \qquad U=\begin{bmatrix} s_0 & \cdots & 0 \\ s_N & \ddots & s_0 \\ 0 & \cdots 0 & s_N \end{bmatrix}$$

be operational matrix and $X_{2N} = [1, \mathrm{t}, \dots, t^{2N}]$, then for derivative term and nonlinear terms in Model (1), we can get

$$\frac{d\mathrm{P_N(t)}}{dt} = \boldsymbol{P}.M.X_N, \quad \frac{d\mathrm{S_N(t)}}{dt} = \boldsymbol{S}.M.X_N, \quad \frac{d\mathrm{Iu_N(t)}}{dt} = \boldsymbol{Iu}.M.X_N,$$

$$\frac{d\mathrm{Ie_N(t)}}{dt} = \boldsymbol{Ie}.M.X_N, \frac{d\mathrm{T_N(t)}}{dt} = \boldsymbol{T}.M.X_N, \frac{d\mathrm{R_N(t)}}{dt} = \boldsymbol{R}.M.X_N,$$

$$\mathrm{S_N(t)}.\mathrm{Iu_N(t)} = X_{2N}.\mathrm{U}.\boldsymbol{Iu},\ \ \mathrm{S_N(t)}.\mathrm{Ie_N(t)} = X_{2N}.\mathrm{U}.\boldsymbol{Ie}, \mathrm{S_N(t)}.\mathrm{T_N(t)} = X_{2N}.\mathrm{U}.\boldsymbol{T}.$$

Now, by substituting these matrix-vector reperstions in model (1), we obtain

$$\begin{aligned}
G1 &:= \boldsymbol{P}.M.X_N - \tau\boldsymbol{S}.X_N - (\mu+\omega)\boldsymbol{P}.X_N - \mu\boldsymbol{S}.X_N \\
G2 &:= \boldsymbol{S}.M.X_N\Lambda + \beta_1 X_{2N}.\mathrm{U}.\boldsymbol{Iu} + \beta_2 X_{2N}.\mathrm{U}.\boldsymbol{Ie} + \beta_3 X_{2N}.\mathrm{U}.\boldsymbol{T} - \omega\boldsymbol{P}.X_N \\
&\qquad + (\tau+\mu)\boldsymbol{S}.X_N \\
G3 &:= \boldsymbol{Iu}.M.X_N - (1-\alpha_1)(\beta_1 X_{2N}.\mathrm{U}.\boldsymbol{Iu} + \beta_2 X_{2N}.\mathrm{U}.\boldsymbol{Ie} + \beta_3 X_{2N}.\mathrm{U}.\boldsymbol{T}) + (\mu+\delta_1 \\
&\qquad + \varphi_1 + \gamma)\boldsymbol{Iu}.X_N \\
G4 &:= \boldsymbol{Ie}.M.X_N - \alpha(\beta_1 X_{2N}.\mathrm{U}.\boldsymbol{Iu} + \beta_2 X_{2N}.\mathrm{U}.\boldsymbol{Ie} + \beta_3 X_{2N}.\mathrm{U}.\boldsymbol{T}) \\
&\qquad + (\mu+\delta_2+\varphi_2)\boldsymbol{Ie}.X_N - \gamma\boldsymbol{Iu}.X_N \\
\mathrm{G5} &:= \boldsymbol{T}.M.X_N - \varphi_1\boldsymbol{Iu}.X_N - \varphi_2\boldsymbol{Ie}.X_N + (\mu+\delta_3+\varepsilon)\boldsymbol{T}.X_N \\
\mathrm{G6} &:= \boldsymbol{R}.M.X_N - \varepsilon\boldsymbol{T}.X_N + \mu\boldsymbol{R}.X_N
\end{aligned}$$

Based on galerkin method Gottlieb and Orszag (1977), 6N algebraic equations is constructed by the following inner product

$$< \mathrm{G1}, t^k >=< \mathrm{G2}, t^k >=< \mathrm{G3}, t^k >=< \mathrm{G4}, t^k >=< \mathrm{G5}, t^k >=< \mathrm{G6}, t^k >= 0,$$

For k= 0,…, N-1. On the other hand, from initial conditions of model (1), yields

$$\mathrm{P_N(0)} = P0, \mathrm{S_N(0)} = S0, \mathrm{Ie_N(0)} = IE, \mathrm{Iu_N(0)} = IU, \mathrm{T_N(0)} = T0, \mathrm{R_N(0)} = R0,$$

By these initial conditions, N+1 algebraic equation, we obtain a system of 6(N+1) equation in which, 6(N+1) unknown coefficients (5). By solving this system, the approximate solutions (4) can be obtained.

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3. Results

Consider a typhoid fever model with

$P(0) = 100, S(0) = 200, Ie(0) = 140, Iu(0) = 120, T(0) = 80, R(0) = 60.$

and parameters values for the model in Table 1. By applying matrix-galerkin method for N=5, we obtain numerical behavior of this system and show in Figures 2-6.

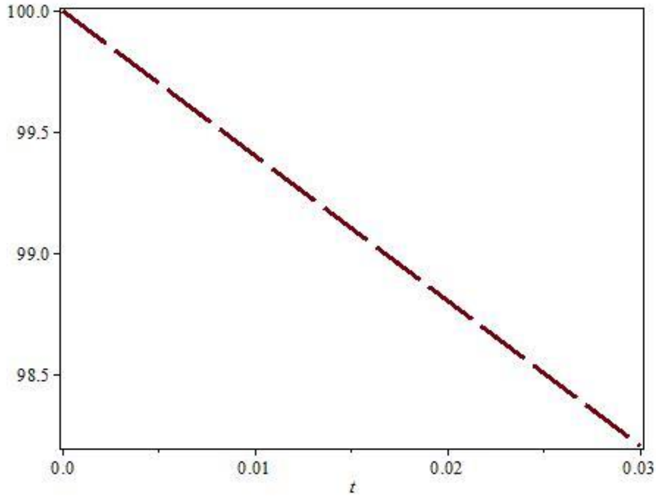

Figure 2. The Behavior of P(t)

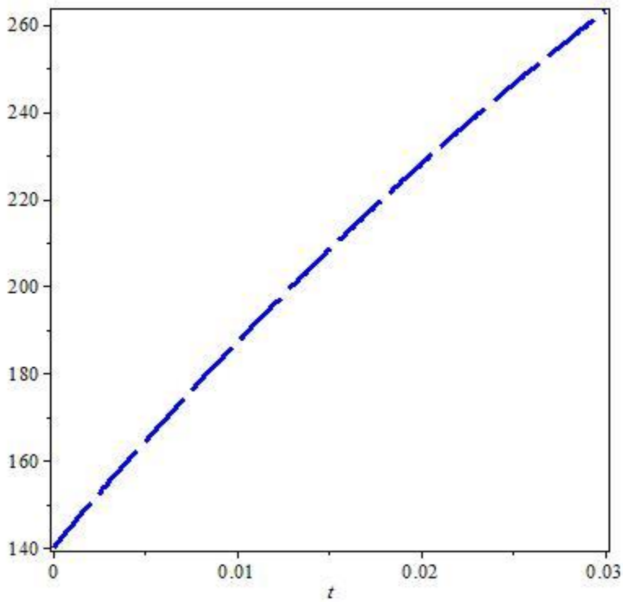

Figure 3. The Behavior of Iu(t)

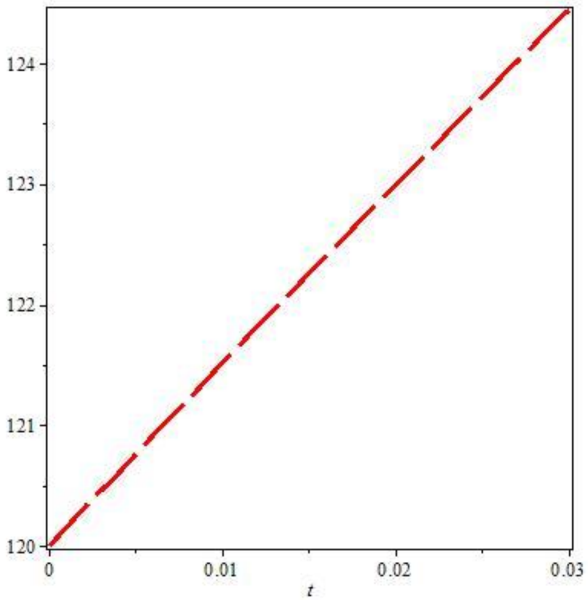

Figure 4. The Behavior of Ie(t)

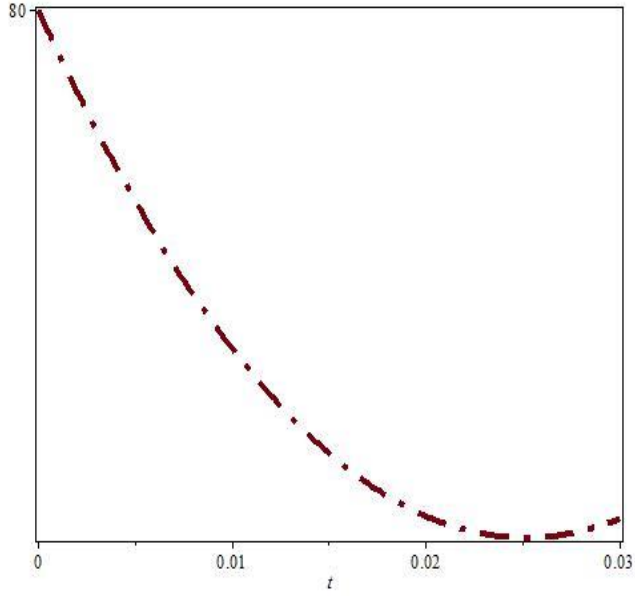

Figure 5. The Behavior of T(t)

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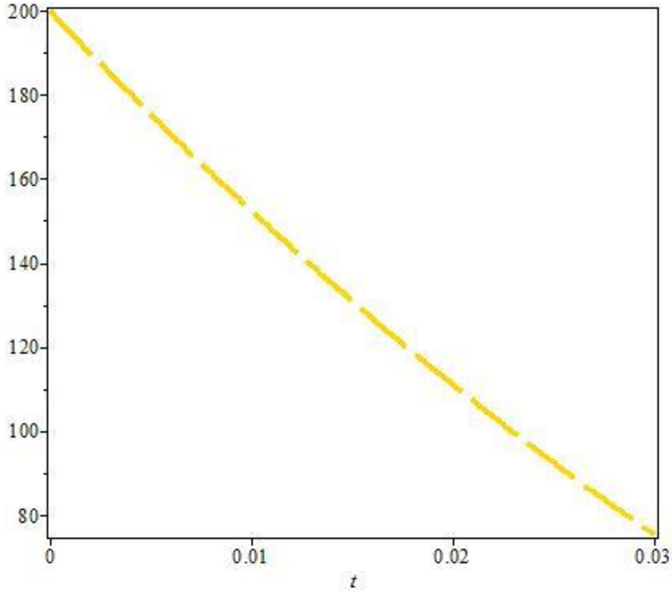

Figure 6. The Behavior of S(t)

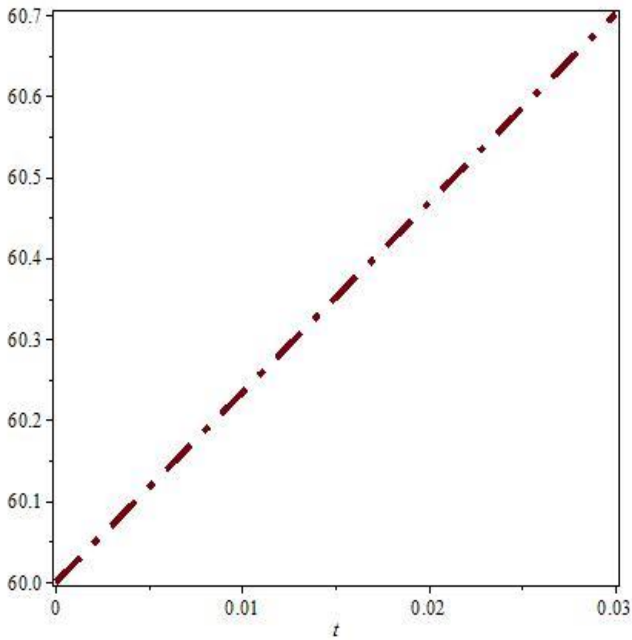

Figure 7. The Behavior of R(t)

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Table 1. Parameters values for the model

Parameter	value
$\delta_1, \delta_2, \delta_3$	0.0075
α	0.0072
τ	0.1
ω	0.5
β_1	1.5
β_2	0.05
β_3	0.05
μ	0.142
γ	0.6
ε	0.4
φ_1	0.04
φ_2	0.3
Λ	200

4. Discussion of Results

The solutions obtained by using Matrix- Galerkin method with given initial conditions is suitable and efficient to conduct the analysis of epidemic models. The results of the simulations were displayed graphically.

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